Discrete Time Arbitrage

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Our key reference is Shreve [2004]. Another useful reference is Brzeźniak and Zastawniak [2005].

Chapter 1

The Binomial No-Arbitrage Pricing Model

1.1 The One-Period Binomial Model

S denotes a stock price; this stock price is governed by the tossing of a coin. Each time a coin is tossed, if a head appears, the price is multiplied by a factor u (the up factor) while if a tail appears the price is multiplied by a factor d (the down factor). No other outcomes are possible.

Let S_0 indicate the stock price at time 0. Furthermore, let $S_1(H)$ indicate the stock price at time 1 if the coin toss yields heads. This occurs with a probability of p_H. Similarly, let $S_1(T)$ indicate the stock price at time 1 if the coin toss yields tails which occurs with probability $p_T = 1 - p_H$.[1] Then

$$u := \frac{S_1(H)}{S_0} \text{ and } d := \frac{S_1(T)}{S_0}$$

and assume without loss of generality that $d < u$.

Let r be the risk-free interest rate, in the sense that 1 deposited in a risk-free bank account at the beginning of the time period will grow to $1 + r$ by the end of the time period. THIS WILL NOT BE STANDARD NOTATION, BUT IS USEFUL FOR NOW.

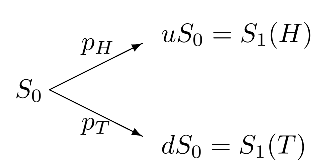

Figure 1.1: One-period binomial tree.

[1] There is no assumption that the coin is fair i.e. that $p_H = p_T = 0.5$. The only requirement is that $p_H, p_T > 0$.

Definition 1.1.1. Arbitrage is a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money.

Definition 1.1.2. A short sale is the sale of something we do not own. This is possible! What is typically done is that we borrow the item from somebody willing to lend it, sell it, buy it back in the market later, and return it to the original owner.

We might do short sales to profit from arbitrages. Usually in reality the owner will ask a fee from you to lend you the stock, but here we will assume that there is no fee.

Proposition 1.1.3. No arbitrage opportunities exist in the one-period binomial model if and only if $0 < d < 1 + r < u$.

Proof. ($\Rightarrow$) Suppose the model is arbitrage free and that $1 + r \geq u > d$ holds. Then, short the stock and invest the proceeds in the bank. At the end of the time period we receive $S_0(1 + r)$ from our investment and we use this to buy back the stock shorted at time 0, which we return. Hence, a cash flow of $S_0(1 + r) - z\, S_0$ (where z is either u or d) which is always greater or equal to 0 (by assumption) and is strictly positive with positive probability. This provides an arbitrage.
Similarly if $u > d \geq 1 + r$ then we borrow from the bank to go long the stock.
($\Leftarrow$) Suppose that $d < 1 + r < u$. Suppose we have 0 initial wealth. There are two ways we could participate in the market.
We can borrow money and buy stock. Suppose we borrow S. At the end our wealth is $-S(1+r) + Sz = S(z - (1 + r))$, where $z \in \{d, u\}$. This is either positive or negative.
We can short stock and deposit the money. Suppose we short one stock. At the end our wealth is $-Sz + S(1 + r) = S(-z + (1 + r))$, where $z \in \{d, u\}$. This is either positive or negative.
In either case, there is no arbitrage. □

Definition 1.1.4. A European call option with strike K has the payoff at time 1 of $V_1 = \max(S_1 - K, 0)$. A European put option with strike K has the payoff at time 1 of $V_1 = \max(K - S_1, 0)$. The option derives its value from the value of the stock, so any such contract is called a derivative.

A derivative is granted (sold) by the one party, known as the short party, and is held (bought) by the other party, known as the long party. A fundamental question is to ask what is the correct premium for this sale?
The correct method of finding the premium is known as replication. Replication involves setting up a portfolio of the stock and risk free asset at $t = 0$ which has the same value as the derivative at time $t = 1$. By no arbitrage the premium of the derivative is equal to the cost of creating the replicating portfolio.
Suppose we want to replicate the derivative whose value V_t at time t for $t = 0, 1$. Let V_0 be the value of the option (and thus of the replicating portfolio as well) and let Δ the number of shares in the replicating portfolio. The final position at time 1 of the replicating strategy is given by

$$X_1 = \Delta S_1 + (1 + r)(V_0 - \Delta S_0) = (1 + r)V_0 + \Delta(S_1 - (1 + r)S_0).$$

In order to replicate the derivative, we require that

$$V_1(H) = (1 + r)V_0 + \Delta\,(S_1(H) - (1 + r)S_0) \tag{1.1}$$

$$V_1(T) = (1 + r)V_0 + \Delta\,(S_1(T) - (1 + r)S_0) \tag{1.2}$$

which is a system of two linear equations in the unknowns V_0 and Δ. Solving (using Cramer's rule, for example) we get

$$\begin{aligned}
\Delta &= \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \\
V_0 &= \frac{V_1(H)[S_1(T) - (1+r)S_0] - V_1(T)[S_1(H) - (1+r)S_0]}{(1+r)(S_1(T) - S_1(H))} \\
&= \frac{V_1(H)[d - (1+r)] - V_1(T)[u - (1+r)]}{(1+r)(d-u)} \\
&= \frac{1}{1+r}\left[\frac{1+r-d}{u-d}V_1(H) + \frac{u-1-r}{u-d}V_1(T)\right] \\
&:= \frac{1}{1+r}\left[q_H V_1(H) + q_T V_1(T)\right]
\end{aligned}$$

where

$$q_H = \frac{1+r-d}{u-d} \text{ and } q_T = \frac{u-1-r}{u-d}$$

are the so-called risk-neutral probabilities. Notice that the real world probabilities never enter the argument!

For another isomorphic treatment see Aristotle [2001].

1.2 The Multi-Period Binomial Model

We now extend §1.1 to multiple periods. The stock price moves up or down N times; the i^{th} move is up or down according to whether the i^{th} toss of the coin is heads or tails. The N tosses form a sample $\omega_1\omega_2\ldots\omega_N$, an arbitrary such word will be denoted ω.

By the multiplicative nature of the moves, the evolution of the stock price can be represented using a multi-period tree.[2]

Any process X which satisfies

$$\begin{aligned}
&X_{n+1}(\omega_1\ldots\omega_n\omega_{n+1}) \\
&= K_n(\omega_1\ldots\omega_n)S(\omega_1\ldots\omega_n\omega_{n+1}) + (1+r)\left[X_n(\omega_1\ldots\omega_n) - K_n(\omega_1\ldots\omega_n)S(\omega_1\ldots\omega_n)\right] \qquad (1.3)
\end{aligned}$$

where $K_n(\omega_1\ldots\omega_n)$ is some number of shares known at time n is called a self-financing portfolio process. The portfolio requires no injection of cash at any stage nor does it disperse any dividends. K_n can be determined by the portfolio owner in any manner whatsoever. However, we could choose K_n in a very particular way, by choosing it to be the delta at every stage. If we do so, we obtain what is called the replicating portfolio, as seen in the following remarkable theorem.

This extends the arbitrage argument of §1.1. It shows that the derivative price process V defined in the obvious naïve way - using risk-neutral discounting all the way through the tree, and then determining the number of stock to hold by the delta formula - is the same process for all times n and all words $\omega_1\ldots\omega_N$, and so is a method of replicating the derivative.

[2]Neither a mathematician nor a gardener would call this a tree, as nodes recombine. Either of them might call this a lattice (a pure mathematician certainly would).

Theorem 1.2.1. Define, by backwards induction, the derivative process

$$V_n(\omega_1 \dots \omega_n) = \frac{1}{1+r}\left[q_H V_{n+1}(\omega_1 \dots \omega_n H) + q_T V_{n+1}(\omega_1 \dots \omega_n T)\right].$$

Let the delta-hedging formula be given by

$$\Delta_n(\omega_1 \dots \omega_n) = \frac{V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)}{S(\omega_1 \dots \omega_n H) - S(\omega_1 \dots \omega_n T)}.$$

Set $X_0 = V_0$ and define the replicating portfolio by forwards induction,

$$\begin{aligned} &X_{n+1}(\omega_1 \dots \omega_n \omega_{n+1}) \\ &= \Delta_n(\omega_1 \dots \omega_n) S(\omega_1 \dots \omega_n \omega_{n+1}) + (1+r)\left[X_n(\omega_1 \dots \omega_n) - \Delta_n(\omega_1 \dots \omega_n) S(\omega_1 \dots \omega_n)\right] \end{aligned} \tag{1.4}$$

Then $X_n(\omega_1 \dots \omega_n) = V_n(\omega_1 \dots \omega_n)$ for all n and all part-words $\omega_1 \dots \omega_n$.

Proof. We proceed by forward induction on n.
The base case, $n = 0$, holds by assumption. Suppose the statement is true for n. We now prove that $X_{n+1}(\omega_1 \dots \omega_n H) = V_{n+1}(\omega_1 \dots \omega_n H)$ holds:

$$\begin{aligned} X_{n+1}(\omega_1 \dots \omega_n H) &= \Delta_n(\omega_1 \dots \omega_n) u S(\omega_1 \dots \omega_n) + (1+r)\left[X_n(\omega_1 \dots \omega_n) - \Delta_n(\omega_1 \dots \omega_n) S(\omega_1 \dots \omega_n)\right] \\ &= (1+r) X_n(\omega_1 \dots \omega_n) + \Delta_n(\omega_1 \dots \omega_n) S(\omega_1 \dots \omega_n)\left[u - (1+r)\right] \\ &= (1+r) V_n(\omega_1 \dots \omega_n) + \left[V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)\right] q_T \\ &= V_{n+1}(\omega_1 \dots \omega_n H). \end{aligned}$$

Similarly it can be shown that $X_{n+1}(\omega_1 \dots \omega_n T) = V_{n+1}(\omega_1 \dots \omega_n T)$. □

Thus, in the multi-period binomial model, every derivative security can be replicated. In this case the market is called complete. When a market is a complete market every derivative security has a unique price - by no arbitrage, it is the cost of setting up the 0^{th} replicating portfolio.

1.3 Computational Considerations

In the most typical simple cases the option is not path dependent - the payoff is determined by the value of the stock price at terminal time, but not by how it got there. In this case a computationally efficient algorithm can be constructed using the stock price tree.
Our representation of trees is not standard but from a computing (that is, writing a program) point of view is unquestionably the best.

Example 1.3.1. Suppose $u = 2$, $d = 0.5$, $r = 0.25$, $N = 5$, $S = 100$. Suppose we have a call option with strike 220.
Note that $q_H = q_T = 0.5$.
We value the option backwards through the tree as in Table 1.1.

But sometimes there is path dependency. In the worst cases there are 2^n terminal nodes to consider - an impossible task if n is even moderately large.
For example, we can consider the case of a lookback put option - this pays off $M - S_N$ where M is the maximum of the stock price along the path that it took.

Spot	0	1	2	3	4	5
0	100	200	400	800	1600	3200
1		50	100	200	400	800
2			25	50	100	200
3				12.5	25	50
4					6.25	12.5
5						3.125
Option	0	1	2	3	4	5
0	60.2112	135.68	302.08	662.4	1424	2980
1		14.848	37.12	92.8	232	580
2			0	0	0	0
3				0	0	0
4					0	0
5						0
Delta	0	1	2	3	4	5
0	0.805547	0.8832	0.949333	0.993333	1	
1		0.494933	0.618667	0.773333	0.966667	
2			0	0	0	
3				0	0	
4					0	

Table 1.1: Pricing a European call option

Example 1.3.2. Let us consider a lookback put option, take $N = 3$, and the same tree parameters as before.
We value the option backwards through the tree as as in Table 1.2.

A trick for handling some path dependent options will be seen in §2.5. This involves having a two-dimensional tree - one for the stock price and another for the path dependent factor. Of course, this is quite computationally intensive. There are advanced and not expensive approximative techniques for handling such derivatives, see [Hull, 2005, 24.4].

1.4 Exercises

1. Shreve [2004] Exercise 1.2 p.20
2. Shreve [2004] Exercise 1.3 p.21
3. Shreve [2004] Exercise 1.4 p.21
4. Shreve [2004] Exercise 1.6 p.21
5. Shreve [2004] Exercise 1.7 p.22. Build a spreadsheet as your solution!
6. Shreve [2004] Exercise 1.8 p.22. For part (ii), use the EXCEL doublelookup function provided.

Spot	0	1	2	3	Max
	100	200	400	800	800
				200	400
			100	200	200
				50	200
		50	100	200	200
				50	100
			25	50	100
				12.5	100
Option	**0**	**1**	**2**	**3**	
	34.4	56	80	0	
				200	
			60	0	
				150	
		30	20	0	
				50	
			55	50	
				87.5	
Delta	**0**	**1**	**2**	**3**	
	0.173333	0.066667	-0.33333		
			-1		
		-0.46667	-0.33333		
			-1		

Table 1.2: Pricing a lookback put

7. (Exam 2008) All options are European.

   Let S be a non-dividend paying stock in a binomial model. Let $p(S,K,T)$ denote a put with maturity T on the stock S with strike K. Let $p_t(S,K,T)$ denote its value at time t. We define and notate a call and a forward, and their values, in a similar way. If there is any doubt, S_t denotes the value of the stock at time t.

   (i) Explain why going long a forward $f(S,K,N)$ and put $p(S,K,N)$ produces the same payoff at N as that of the call $c(S,K,N)$.

   (ii) Using risk neutral valuation deduce that $c_n(S,K,N) = f_n(S,K,N) + p_n(S,K,N)$ for every $n \leq N$.

   (iii) Show that $f_n(S,K,N) = S_n - \frac{K}{(1+r)^{N-n}}$ for every $n \leq N$.

   (iv) A chooser option is an option bought at time 0 that gives the long party the right to receive at some fixed time n, $0 < n < N$, either a call or a put for expiry N. The long party only has to make their choice at time n.

Let $ch(S,K,T)$ and $ch_t(S,K,T)$ be similar notation for chooser options to the previous for vanilla options. Write down the value $ch_n(S,K,n)$.

(v) Show that $ch_n(S,K,n) = p_n(S,K,N) + \max\left(0, S_n - \frac{K}{(1+r)^{N-n}}\right)$.

(vi) Thus write down the value of $ch_0(S,K,n)$.

Chapter 2

Probability Theory on Coin Toss Space

2.1 Finite Probability Spaces

Definition 2.1.1. A sample space Ω is a set with finitely many elements.
A probability measure $\mathbb{P}$ is a function

$$\mathbb{P} : \Omega \to [0,1] \text{ such that } \sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

We extend the definition as follows

$$\mathbb{P} : 2^{\Omega} \to [0,1] \text{ as } \mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega).$$

Example 2.1.2. Let Ω be the space of N coin tosses from Chapter 1. The elements of Ω are the words $\omega_1\omega_2 \dots \omega_N$ where each ω_i is H or T, NOT the set $\{\omega_1, \ \dots, \ \omega_N\}$. Let $\mathbb{P}(\omega) = p_H^{\#H(\omega)} p_T^{\#T(\omega)}$ where $\#H(\omega)/\#T(\omega)$ is the number of heads/tails in the ω. It is a good idea to call it a word - the order of the letters are important.

Remark 2.1.3. It is important that the number of elements in Ω is finite. Else we have definitional problems with the measure on infinite sets.

2.2 Random Variables, Distributions and Expectations

Definition 2.2.1. A random variable on Ω is a function $X : \Omega \to \mathbb{R}$ (or on $X : \Omega \to [-\infty, \infty]$).

Example 2.2.2. Consider the tree from §1.3. Let X be the stock price after two tosses. Then

$$\begin{aligned} X(HH\omega_3 \dots \omega_N) &= 400 \\ X(HT\omega_3 \dots \omega_N) &= 100 \\ X(TH\omega_3 \dots \omega_N) &= 100 \\ X(TT\omega_3 \dots \omega_N) &= 25 \end{aligned}$$

for every $\omega_3 \dots \omega_N$.

Note that $\mathbb{P}$ is not relevant for the definition of the random variable. We could have another measure $\mathbb{Q}$ and the random variable is unchanged.

Definition 2.2.3. Suppose X is a random variable on a sample space Ω with probability measure $\mathbb{P}$. The distribution of X is

$$d(X) : \mathbb{R} \to \mathbb{R} : x \to \mathbb{P}\left(\{\omega \in \Omega : X(\omega) = x\}\right) \equiv \mathbb{P}(X(\omega) = x) \equiv \mathbb{P}(X = x)$$

Example 2.2.4. Start with the measure $\mathbb{P}$ where $p_H = p_T = \frac{1}{2}$. X is as before

$$\begin{aligned} \mathbb{P}(X = 400) &= \frac{1}{4} \\ \mathbb{P}(X = 100) &= \frac{1}{2} \\ \mathbb{P}(X = 25) &= \frac{1}{4}. \end{aligned}$$

Example 2.2.5. Now consider the measure $\mathbb{Q}$ where $p_H = \frac{3}{5}$, $p_T = \frac{2}{5}$.

$$\begin{aligned} \mathbb{Q}(X = 400) &= \frac{9}{25} \\ \mathbb{Q}(X = 100) &= \frac{12}{25} \\ \mathbb{Q}(X = 25) &= \frac{4}{25}. \end{aligned}$$

Definition 2.2.6. Let X be a random variable. The expected value of X with respect to the measure $\mathbb{P}$ is

$$\mathbb{E}^{\mathbb{P}}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega).$$

We just write $\mathbb{E}[X]$ if the measure is clear.

Definition 2.2.7. The variance of X with respect to the measure $\mathbb{P}$ is

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}}\left[\left(X - \mathbb{E}^{\mathbb{P}}[X]\right)^2\right].$$

Lemma 2.2.8.

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}}[X^2] - \left(\mathbb{E}^{\mathbb{P}}[X]\right)^2.$$

Proof. Exercise. □

Theorem 2.2.9 (Jensen's Inequality)**.** X be a random variable on a finite probability space. Let φ be a convex function. Then

$$\mathbb{E}^{\mathbb{P}}\left[\varphi(X)\right] \geq \varphi\left(\mathbb{E}^{\mathbb{P}}[X]\right).$$

Proof. For any $x \in \mathbb{R}$

$$\varphi(x) = \sup\{l(x) : l \text{ is affine and } \varphi(y) \geq l(y)\ \forall y \in \mathbb{R}\}.$$

Suppose $l(y) = my + c$ is such a test function.

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\varphi(X)\right] &= \sum_{\omega\in\Omega} \varphi(X(\omega))\mathbb{P}(\omega) \\
&\geq \sum_{\omega\in\Omega} l(X(\omega))\mathbb{P}(\omega) \\
&= \sum_{\omega\in\Omega} (mX(\omega) + c)\mathbb{P}(\omega) \\
&= m \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\omega) + c \sum_{\omega\in\Omega} \mathbb{P}(\omega) \\
&= m\mathbb{E}^{\mathbb{P}}[X] + c \\
&= l\left(\mathbb{E}^{\mathbb{P}}[X]\right).
\end{aligned}$$

So $\mathbb{E}^{\mathbb{P}}\left[\varphi(X)\right] \geq \varphi\left(\mathbb{E}^{\mathbb{P}}[X]\right)$ as l was an arbitrary test function. □

2.3 Conditional Expectations

Definition 2.3.1 (Conditional Expectation)**.** The conditional expectation of X based on information at time n is given by

$$\mathbb{E}^{\mathbb{P}}_n[X](\omega_1 \dots \omega_n) = \sum_{\omega_{n+1},\dots,\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} X(\omega_1 \dots \omega_n \omega_{n+1} \dots \omega_N). \tag{2.1}$$

Theorem 2.3.2. (i) (Linearity) For constants a and b, $\mathbb{E}_n[aX + bY] = a\mathbb{E}_n[X] + b\mathbb{E}_n[Y]$.

(ii) (Taking out what is known) If X only depends on the first n coin tosses then $\mathbb{E}_n[XY] = X\mathbb{E}_n[Y]$.

(iii) (Iterated conditioning / Tower property) For $0 \leq n \leq m \leq N$, $\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X]$.

(iv) (Independence) X depends only on coin tosses $n + 1, \dots N \Rightarrow \mathbb{E}_n[X] = \mathbb{E}[X]$.

(v) (Conditional Jensen's inequality) $\varphi(x)$ a convex function of dummy variable $x \Rightarrow \mathbb{E}_n[\varphi(X)] \geq \varphi(\mathbb{E}_n[X])$.

Proof. (i)

$$
\begin{aligned}
&\mathbb{E}_n[aX+bY](\omega_1\dots\omega_n)\\
&=\sum_{\omega_{n+1}\dots\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} (aX(\omega_1\dots\omega_N)+bY(\omega_1\dots\omega_N))\\
&=a\sum_{\omega_{n+1}\dots\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} X(\omega_1\dots\omega_N)\\
&+b\sum_{\omega_{n+1}\dots\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} Y(\omega_1\dots\omega_N)\\
&=a\mathbb{E}_n[X](\omega_1\dots\omega_n)+b\mathbb{E}_n[Y](\omega_1\dots\omega_n).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\mathbb{E}_n[XY](\omega_1\dots\omega_n) &= \sum_{\omega_{n+1}\dots\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} X(\omega_1\dots\omega_n)Y(\omega_1\dots\omega_N)\\
&= X(\omega_1\dots\omega_n)\sum_{\omega_{n+1}\dots\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} Y(\omega_1\dots\omega_N)\\
&= X\mathbb{E}_n[Y](\omega_1\dots\omega_n).
\end{aligned}
$$

(iii) Let $Z=\mathbb{E}_m[X]$ so Z depends on $\omega_1\dots\omega_m$ only.

$$
\begin{aligned}
&\mathbb{E}_n[\mathbb{E}_m[X]](\omega_1\dots\omega_n)\\
=&\mathbb{E}_n[Z](\omega_1\dots\omega_n)\\
=&\sum_{\omega_{n+1}\dots\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} Z(\omega_1\dots\omega_n\omega_{n+1}\dots\omega_m)\\
=&\sum_{\omega_{n+1}\dots\omega_m} p_H^{\#H(\omega_{n+1},\dots,\omega_m)} p_T^{\#T(\omega_{n+1},\dots,\omega_m)} Z(\omega_1\dots\omega_m)\\
&\times\sum_{\omega_{m+1}\dots\omega_N} p_H^{\#H(\omega_{m+1},\dots,\omega_N)} p_T^{\#T(\omega_{m+1},\dots,\omega_N)}\\
=&\sum_{\omega_{n+1}\dots\omega_m} p_H^{\#H(\omega_{n+1},\dots,\omega_m)} p_T^{\#T(\omega_{n+1},\dots,\omega_m)} Z(\omega_1\dots\omega_m)\\
=&\sum_{\omega_{n+1}\dots\omega_m} p_H^{\#H(\omega_{n+1},\dots,\omega_m)} p_T^{\#T(\omega_{n+1},\dots,\omega_m)}\\
&\times\sum_{\omega_{m+1},\dots,\omega_N} p_H^{\#H(\omega_{m+1},\dots,\omega_N)} p_T^{\#T(\omega_{m+1},\dots,\omega_N)} X(\omega_1\dots\omega_N)\\
=&\sum_{\omega_{n+1},\dots,\omega_N} p_H^{\#H(\omega_{n+1},\dots,\omega_N)} p_T^{\#T(\omega_{n+1},\dots,\omega_N)} X(\omega_1\dots\omega_N)\\
=&\mathbb{E}_n[X](\omega_1\dots\omega_n).
\end{aligned}
$$

(iv)

$$\begin{aligned}
\mathbb{E}_n[X](\omega_1 \ldots \omega_n) = & \sum_{\omega_{n+1}\ldots\omega_N} p_H^{\#H(\omega_{n+1},\ldots,\omega_N)} p_T^{\#T(\omega_{n+1},\ldots,\omega_N)} X(\omega_{n+1}\ldots\omega_N) \\
= & \sum_{\omega_1\ldots\omega_n} p_H^{\#H(\omega_1,\ldots,\omega_n)} p_T^{\#T(\omega_1,\ldots,\omega_n)} \\
& \times \sum_{\omega_{n+1}\ldots\omega_N} p_H^{\#H(\omega_{n+1},\ldots,\omega_N)} p_T^{\#T(\omega_{n+1},\ldots,\omega_N)} X(\omega_{n+1}\ldots\omega_N) \\
= & \sum_{\omega_1\ldots\omega_N} p_H^{\#H(\omega_1,\ldots,\omega_N)} p_T^{\#T(\omega_1,\ldots,\omega_N)} X(\omega_{n+1}\ldots\omega_N) \\
= & \mathbb{E}[X].
\end{aligned}$$

(v)

$$\begin{aligned}
\mathbb{E}_n\left[\varphi(X)\right](\omega_1\ldots\omega_n) &= \sum_{\omega\in\Omega} p_H^{\#H(\omega_{n+1},\ldots,\omega_N)} p_T^{\#T(\omega_{n+1},\ldots,\omega_N)} \varphi(X(\omega_1\ldots\omega_N)) \\
&\geq \sum_{\omega\in\Omega} p_H^{\#H(\omega_{n+1},\ldots,\omega_N)} p_T^{\#T(\omega_{n+1},\ldots,\omega_N)} l(X(\omega_1\ldots\omega_N)) \\
&= \mathbb{E}_n\left[l(X)\right](\omega_1\ldots\omega_n) \\
&= l\left(\mathbb{E}_n[X]\right)(\omega_1\ldots\omega_n)
\end{aligned}$$

for an arbitrary test function l.

□

2.4 Martingales

Definition 2.4.1. Let $\underline{M} = M_0, M_1, \ldots, M_N$ be a sequence of random variables. Then $\underline{M}$ is adapted if each M_k is dependent on the first k coin tosses only.

Example 2.4.2. The hedging process Δ is adapted.

Example 2.4.3. The wealth process X/V is adapted.

Definition 2.4.4 (Martingale, submartingale and supermartingale)**.** Let $\underline{M}$ be adapted, then $\underline{M}$ is a

- $\mathbb{P}$-martingale if $M_n = \mathbb{E}_n^{\mathbb{P}}[M_{n+1}]$
- $\mathbb{P}$-submartingale if $M_n \leq \mathbb{E}_n^{\mathbb{P}}[M_{n+1}]$
- $\mathbb{P}$-supermartingale if $M_n \geq \mathbb{E}_n^{\mathbb{P}}[M_{n+1}]$

for $n = 0, 1, \ldots, N-1$.

Exercise 2.4.1. Show that if $\langle M_1, \ldots, M_n\rangle$ is a martingale then $M_n = \mathbb{E}_n^{\mathbb{P}}[M_m]$ for all $m > n$.

Exercise 2.4.2. Show that if $\langle M_1, \ldots, M_n\rangle$ is a martingale then $\langle M_1^2, \ldots, M_n^2\rangle$ is a submartingale.

Exercise 2.4.3. Show that $S_k = \frac{1}{1+r}\mathbb{E}_k^{\mathbb{Q}}\left[S_{k+1}\right]$ where $\mathbb{Q}$ denotes the risk-neutral measure.

This generalises as follows:

Theorem 2.4.5. The discounted stock price S is a martingale under the risk neutral measure $\mathbb{Q}$.

Proof.

$$\begin{aligned}
\mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^{n+1}}\frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^n}\frac{1}{1+r}\mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^n}\frac{q_H u + q_T d}{1+r} \\
&= \frac{S_n}{(1+r)^n}.
\end{aligned}$$

□

(1.4) shows that $X_k = \frac{1}{1+r}\mathbb{E}_k^{\mathbb{Q}}\left[X_{k+1}\right]$. This generalises as follows:

Theorem 2.4.6. The discounted wealth process of any self-financing strategy X is a martingale under the risk-neutral measure $\mathbb{Q}$.

Proof. From (1.3) we have that

$$\begin{aligned}
X_{n+1} &= \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) \\
\Rightarrow \mathbb{E}_n^{\mathbb{Q}}\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \mathbb{E}_n^{\mathbb{Q}}\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \mathbb{E}_n^{\mathbb{Q}}\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \mathbb{E}_n^{\mathbb{Q}}\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \Delta_n \mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{X_n}{(1+r)^n}.
\end{aligned}$$

□

Theorem 2.4.7 (Risk-Neutral Valuation)**.**

$$V_0 = \frac{1}{(1+r)^N}\mathbb{E}_0^{\mathbb{Q}}\left[V_N\right].$$

where $\mathbb{Q}$ denotes the risk-neutral measure.

Proof. This follows from Theorems 1.2.1 and 2.4.6. The replication portfolio is self financing. □

2.5 Markov Processes

Definition 2.5.1 (Markov Process)**.** Let $X_0, X_1, \ldots, X_N$ be any adapted process. If for every n and for every function f there exists a function g such that

$$\mathbb{E}_n^{\mathbb{R}}\left[f(X_{n+1})\right] = g(X_n)$$

then we say X is a Markov process under the measure $\mathbb{R}$.

In the following three examples, the measure is either $\mathbb{P}$ or $\mathbb{Q}$, $\mathbb{E}$ and $\mathbb{E}_n$ are with respect to that measure, and p_H, p_T refer to the up and down probabilities under that measure.

Example 2.5.2. Suppose X is the stock price process S, then

$$\mathbb{E}_n\left[f\left(X_{n+1}\right)\right] = p_H f\left(uS_n\right) + p_T f\left(dS_n\right).$$

So $g(x) = p_H f(ux) + p_T f(dx)$ and hence X is Markov.

Example 2.5.3. Suppose X is the maximum to date process M. Then

$$\mathbb{E}_n\left[f\left(M_{n+1}\right)\right] = \begin{cases} p_H f\left(uM_n\right) + p_T f\left(M_n\right), & \text{if } S_n = M_n \\ f\left(M_n\right), & \text{if } S_n \neq M_n \end{cases}$$

which is not a function of M_n only. So the process is not Markov.

Example 2.5.4. Let X be the bivariate process (M, S). Then

$$\begin{aligned} \mathbb{E}_n\left[f\left(M_{n+1}, S_{n+1}\right)\right] &= \begin{cases} p_H f\left(uM_n, uS_n\right) + p_T f\left(M_n, dS_n\right), & \text{if } S_n = M_n \\ p_H f\left(M_n, uS_n\right) + p_T f\left(M_n, dS_n\right), & \text{if } S_n \neq M_n \end{cases} \\ &= g\left(M_n, S_n\right) \end{aligned}$$

where

$$\begin{aligned} g(m, s) &:= \begin{cases} p_H f(um, us) + p_T f(m, ds), & \text{if } s = m \\ p_H f(m, us) + p_T f(m, ds), & \text{if } s \neq m \end{cases} \\ &= p_H f(m \vee us, us) + p_T f(m, ds). \end{aligned}$$

Thus X is Markov.

Once we 'upgrade' a non-Markovian process into a Markovian one, we can price off of a tree where at least some recombination is present.

Example 2.5.5. Now let us consider the lookback put of §1.3. Let $f(M_n, S_n)$ be the option value at time n if the stock price is S_n and the maximum so far is M_n. Then

$$\begin{aligned} f(M_n, S_n) &= \frac{1}{1+r}\mathbb{E}_n^{\mathbb{Q}}\left[f(M_{n+1}, S_{n+1})\right] \\ &= \frac{1}{1+r} g(M_n, S_n) \\ &= \frac{1}{1+r}\left[q_H f(M_n \vee uS_n, uS_n) + q_T f(M_n, dS_n)\right] \end{aligned}$$

Now $f(M_N, S_N) = M_N - S_N$. We can value the option using this backwards induction formula on a two-dimensional tree for (M, S).

2.6 Exercises

1. Shreve [2004] Exercise 2.1 p.54

2. Shreve [2004] Exercise 2.2 p.55

		S						
n =3		12.5	25	50	100	200	400	800
M	100	87.5	75	50	0	-100	-300	-700
	200	187.5	175	150	100	0	-200	-600
	400	387.5	375	350	300	200	0	-400
	800	787.5	775	750	700	600	400	0
n=2								
M	100		55	30	20	0	-40	
	200		135	110	60	40	0	
	400		295	270	220	120	80	
n=1								
M	100			30	28	40		
	200			78	60	56		
n=0								
M	100				34.4			

Table 2.1: The value of the lookback put option at times $n = 0, 1, 2, 3$ given the different possible values of the maximum M and corresponding stock price S.

3. Exercise 2.4.1

4. Exercise 2.4.2

5. Exercise 2.4.3

6. Shreve [2004] Exercise 2.3 p.55

7. Shreve [2004] Exercise 2.4 p.55

8. Shreve [2004] Exercise 2.5 p.55

9. Shreve [2004] Exercise 2.6 p.56

10. Shreve [2004] Exercise 2.7 p.56 Hint: think of a bet which pays off on every toss of the coin, but unusually depends in some way on the history of the tosses made so far.

11. Shreve [2004] Exercise 2.10 p.58

12. Shreve [2004] Exercise 2.11 p.58

13. Shreve [2004] Exercise 2.12 p.59

14. Shreve [2004] Exercise 2.13(ii) p.59

15. Shreve [2004] Exercise 2.14 p.60

16. (Exam 2008) Consider an infinite sequence of independent coin tosses of a fair coin. Consider a process X_i ($i \geq 2$) that pays 1 if we have the $(i-1)^{th}$ and i^{th} tosses are both heads or both tails, and pays -1 if they are different. Consider the process M which is the aggregate winnings i.e. where $M_1 = 0$ and $M_n = \sum_{i=2}^{n} X_i$ for $n \geq 2$.

Now consider another game where the process Y_i ($i \geq 3$) pays 0 if the $(i-2)^{th}$ and $(i-1)^{th}$ tosses are the same. If they are different then the payoff is 1 if the $(i-2)^{th}$ and i^{th} tosses are the same and the payoff is -1 if they are different. Consider the process N which is the aggregate winnings i.e. where $N_1 = N_2 = 0$ and $N_n = \sum_{i=3}^{n} Y_i$ for $n \geq 3$.

(i) Prove that M is a martingale.

(ii) Prove that M is Markov.

(iii) Prove that N is a martingale.

(iv) Prove that N is not Markov.

(v) How would N be efficiently augmented to become a Markov process? You just need to write down an answer (there are several), do not prove it.

Chapter 3

State Prices

3.1 Change of Numéraire

Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures with $\mathbb{Q}(\omega) \neq 0$ for all ω. The Radon-Nikodým derivative is given by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega) = \frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)}.$$

If $\mathbb{Q}(\omega) \neq 0 \neq \mathbb{P}(\omega)$ for all ω then we say $\mathbb{Q}$ and $\mathbb{P}$ are equivalent (we write $\mathbb{Q} \sim \mathbb{P}$). It means that the measures agree on what is possible.

Example 3.1.1. Let the actual coin toss probabilities be p_H, p_T with measure $\mathbb{P}$ and the risk-neutral probabilities be q_H, q_T with measure $\mathbb{Q}$. Then

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega) = \left(\frac{p_H}{q_H}\right)^{\#H(\omega_1,\ldots,\omega_N)} \left(\frac{p_T}{q_T}\right)^{\#T(\omega_1,\ldots,\omega_N)}.$$

Also,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \left(\frac{q_H}{p_H}\right)^{\#H(\omega_1,\ldots,\omega_N)} \left(\frac{q_T}{p_T}\right)^{\#T(\omega_1,\ldots,\omega_N)}.$$

Example 3.1.2. Let $p_H = \frac{2}{3}$, $p_T = \frac{1}{3}$, $q_H = \frac{1}{2} = q_T$ and $N = 3$. Then

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(HHH) = \left(\frac{2/3}{1/2}\right)^3 = \frac{64}{27}$$
$$\frac{d\mathbb{P}}{d\mathbb{Q}}(HHT) = \left(\frac{2/3}{1/2}\right)^2 \left(\frac{1/3}{1/2}\right) = \frac{32}{27}$$
$$\frac{d\mathbb{P}}{d\mathbb{Q}}(HTT) = \left(\frac{2/3}{1/2}\right) \left(\frac{1/3}{1/2}\right)^2 = \frac{16}{27}$$
$$\frac{d\mathbb{P}}{d\mathbb{Q}}(TTT) = \left(\frac{1/3}{1/2}\right)^3 = \frac{8}{27}$$

Theorem 3.1.3. Suppose $\mathbb{Q} \sim \mathbb{P}$, then

(i) $\mathbb{Q}\left(\frac{d\mathbb{P}}{d\mathbb{Q}} > 0\right) = 1$.

(ii) $\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\right] = 1$.

(iii) $\mathbb{E}^{\mathbb{P}}[Y] = \mathbb{E}^{\mathbb{Q}}\left[Y\frac{d\mathbb{P}}{d\mathbb{Q}}\right]$ for any random variable Y.

Proof. (i) $\left\{\omega \in \Omega : \frac{d\mathbb{P}}{d\mathbb{Q}} = 0\right\} = \emptyset$.

(ii)

$$\begin{aligned}\sum_{\omega\in\Omega} \frac{d\mathbb{P}}{d\mathbb{Q}}(\omega)\mathbb{Q}(\omega) &= \sum_{\omega\in\Omega} \frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)}\mathbb{Q}(\omega) \\ &= \sum_{\omega\in\Omega} \mathbb{P}(\omega) = 1.\end{aligned}$$

(iii)

$$\begin{aligned}\sum_{\omega\in\Omega} Y(\omega)\mathbb{P}(\omega) &= \sum_{\omega\in\Omega} Y(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)}\mathbb{Q}(\omega) \\ &= \sum_{\omega\in\Omega} Y(\omega)\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega)\mathbb{Q}(\omega) \\ &= \mathbb{E}^{\mathbb{Q}}\left[Y\frac{d\mathbb{P}}{d\mathbb{Q}}\right].\end{aligned}$$

□

Example 3.1.4 (Arrow-Debreu Price)**.** Fix $\alpha = \alpha_1 \dots \alpha_N$. Consider the security payoff

$$V_N(\omega) = \begin{cases} 1 \text{ if } \omega = \alpha \\ 0 \text{ otherwise} \end{cases}$$

Then

$$\begin{aligned}V_0 &= \frac{1}{(1+r)^N}\mathbb{E}^{\mathbb{Q}}\left[V_N(\omega)\right] \\ &= \frac{1}{(1+r)^N} q_H^{\#H(\alpha)} q_T^{\#T(\alpha)} \\ &= \frac{1}{(1+r)^N}\mathbb{Q}(\alpha).\end{aligned}$$

For any derivative

$$V_0 = \frac{1}{(1+r)^N}\sum_{\omega\in\Omega} V_N(\omega)\mathbb{Q}(\omega).$$

where $V_N(\omega)$ is the payoff if ω occurs (it might be different from 1).

Definition 3.1.5 (State Price Density Random Variable)**.** The state price density random variable is given by

$$\zeta(\omega) := \frac{1}{(1+r)^N}\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \frac{1}{(1+r)^N}\frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}.$$

Thus

$$V_0 = \sum_{\omega\in\Omega} V_N(\omega)\zeta(\omega)\mathbb{P}(\omega).$$

3.2 Radon-Nikodým Derivative Process

Theorem 3.2.1. Let $X = X(\omega_1 \dots \omega_N)$ be a random variable. Then $\mathbb{E}_n^{\mathbb{P}}[X]$ is a martingale.

Proof.

$$\mathbb{E}_n^{\mathbb{P}}\left[\mathbb{E}_{n+1}^{\mathbb{P}}[X]\right] = \mathbb{E}_n^{\mathbb{P}}[X].$$

□

As formulated, the Radon-Nikodým derivative is not very useful: the definition of $\frac{d\mathbb{P}}{d\mathbb{Q}}$ or $\frac{d\mathbb{Q}}{d\mathbb{P}}$ required all N coin tosses by which time it is too late to do anything useful with it. To get useful information we want to estimate (find the expectation of) the Radon-Nikodým derivative based on information at time $n < N$.

Definition 3.2.2. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures. Then

$$Z_n = \mathbb{E}_n^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

is called the Radon-Nikodým derivative process.

Lemma 3.2.3. Suppose Y is a random variable depending only the first n coin tosses. Then

$$\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{P}}[Z_n Y].$$

Proof.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[Y] &= \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}Y\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}_n^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}Y\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Y\mathbb{E}_n^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}[YZ_n]
\end{aligned}$$

□

In the following we extend Theorem 3.1.3 (ii).

Remark 3.2.4.

$$Z_n(\omega_1 \dots \omega_n) = \left(\frac{q_H}{p_H}\right)^{\#H(\omega_1,\dots,\omega_n)}\left(\frac{q_T}{p_T}\right)^{\#T(\omega_1,\dots,\omega_n)}.$$

Proof. Let $Y = \begin{cases} 1 \text{ if } \omega_1 \dots \omega_n \text{ is reached} \\ 0 \text{ if not} \end{cases}$. Then

$$\mathbb{E}^{\mathbb{Q}}[Y] = q_H^{\#H(\omega_1,\dots,\omega_n)} q_T^{\#T(\omega_1,\dots,\omega_n)}$$
$$\mathbb{E}^{\mathbb{P}}[YZ_n] = p_H^{\#H(\omega_1,\dots,\omega_n)} p_T^{\#T(\omega_1,\dots,\omega_n)} Z_n(\omega_1 \dots \omega_n).$$

□

Example 3.2.5. Suppose $N = 3$ and $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Firstly we find

$$Z_3(HHH) = \left(\frac{1/2}{2/3}\right)^3 = \frac{27}{64}$$
$$Z_3(HHT) = \left(\frac{1/2}{2/3}\right)^2 \left(\frac{1/2}{1/3}\right) = \frac{27}{32} = Z_3(HTH) = Z_3(THH)$$
$$Z_3(HTT) = \left(\frac{1/2}{2/3}\right) \left(\frac{1/2}{1/3}\right)^2 = \frac{27}{16} = Z_3(THT) = Z_3(TTH)$$
$$Z_3(TTT) = \left(\frac{1/2}{2/3}\right)^3 = \frac{27}{8}$$

So

$$Z_2(HH) = \frac{2}{3}Z_3(HHH) + \frac{1}{3}Z_3(HHT) = \frac{9}{16}$$
$$Z_2(HT) = \frac{2}{3}Z_3(HTH) + \frac{1}{3}Z_3(HTT) = \frac{9}{8} = Z_2(TH)$$
$$Z_2(TT) = \frac{2}{3}Z_3(TTH) + \frac{1}{3}Z_3(TTT) = \frac{9}{4}.$$

and

$$Z_1(H) = \frac{4}{9}Z_3(HHH) + \frac{4}{9}Z_3(HTH) + \frac{1}{9}Z_3(HTT) = \frac{3}{4}$$
$$Z_1(T) = \frac{4}{9}Z_3(THH) + \frac{4}{9}Z_3(TTH) + \frac{1}{9}Z_3(TTT) = \frac{3}{2}$$
$$\text{or } Z_1(H) = \mathbb{E}_1^{\mathbb{P}}[Z_2|\omega_1 = H] = \frac{2}{3}Z_2(HH) + \frac{1}{3}Z_2(HT) = \frac{3}{4}$$
$$Z_1(T) = \mathbb{E}_1^{\mathbb{P}}[Z_2|\omega_1 = T] = \frac{2}{3}Z_2(TH) + \frac{1}{3}Z_2(TT) = \frac{3}{2}$$

Finally

$$Z_0 = \frac{2}{3}Z_1(H) + \frac{1}{3}Z_1(T) = 1$$

Lemma 3.2.6. Let $n < m$. Let Y be a random variable depending on the first m coin tosses only. Then

$$\mathbb{E}_n^{\mathbb{Q}}[Y] = \frac{1}{Z_n}\mathbb{E}_n^{\mathbb{P}}[Z_m Y].$$

Proof.

$$
\begin{aligned}
&\mathbb{E}_n^{\mathbb{Q}}[Y](\omega_1 \dots \omega_n) \\
&= \sum_{\omega_{n+1},\dots,\omega_m} q_H^{\#H(\omega_{n+1},\dots,\omega_m)} q_T^{\#T(\omega_{n+1},\dots,\omega_m)} Y(\omega_1 \dots \omega_n \omega_{n+1} \dots \omega_m) \\
&= \left(\frac{p_H}{q_H}\right)^{\#H(\omega_1,\dots,\omega_n)} \left(\frac{p_T}{q_T}\right)^{\#T(\omega_1,\dots,\omega_n)} \sum_{\omega_{n+1},\dots,\omega_m} \frac{q_H^{\#H(\omega_1,\dots,\omega_m)}}{p_H^{\#H(\omega_1,\dots,\omega_n)}} \frac{q_T^{\#T(\omega_1,\dots,\omega_m)}}{p_T^{\#T(\omega_1,\dots,\omega_n)}} Y(\omega_1 \dots \omega_n \omega_{n+1} \dots \omega_m) \\
&= \frac{1}{Z(\omega_1 \dots \omega_n)} \sum_{\omega_{n+1},\dots,\omega_m} p_H^{\#H(\omega_{n+1},\dots,\omega_m)} p_T^{\#T(\omega_{n+1},\dots,\omega_m)} \\
&\qquad \left(\frac{q_H}{p_H}\right)^{\#H(\omega_1,\dots,\omega_m)} \left(\frac{q_T}{p_T}\right)^{\#T(\omega_1,\dots,\omega_m)} Y(\omega_1 \dots \omega_n \omega_{n+1} \dots \omega_m) \\
&= \frac{1}{Z(\omega_1 \dots \omega_n)} \mathbb{E}_n^{\mathbb{P}}\left[Z_m Y\right](\omega_1 \dots \omega_n).
\end{aligned}
$$

□

3.3 Utility Based Pricing

Definition 3.3.1 (Concave Function)**.** A function U is called strictly concave if for every $x, y \in \mathbb{R}$ and $\alpha \in (0,1)$ we have that

$$U\left(\alpha x + (1-\alpha)y\right) > \alpha U(x) + (1-\alpha)U(y).$$

The twice differentiable strictly concave functions are exactly those for which $U'' < 0$.

Example 3.3.2. The ln-function is concave.

Definition 3.3.3. A utility function is an increasing strictly concave function.

The problem faced by rational investors is: given X_0 and a utility function U, find a process (adapted and self-financing) that maximises $\mathbb{E}^{\mathbb{P}}\left[U\left(X_N\right)\right]$. Here $\mathbb{P}$ is the real-world measure.
Equivalent to this is: given X_0 and U find a random variable X such that $X_0 = \frac{1}{(1+r)^N}\mathbb{E}^{\mathbb{Q}}\left[X_N\right]$ and $\mathbb{E}^{\mathbb{P}}\left[U\left(X_N\right)\right]$ is maximised. This really reduces the problem to finding X_N, because having done so, we can easily find X_0 using Theorem 1.2.1.

Example 3.3.4. Consider a two-period model and the ln-utility function with $p_H = \frac{2}{3}, p_T = \frac{1}{3}$ and $q_H = \frac{1}{2} = q_T$. Furthermore, let $X_0 = 4$ and $r = \frac{1}{4}$.
Let $x_1 = X(HH)$, $x_2 = X(HT)$, $x_3 = X(TH)$, $x_4 = X(TT)$. Then the problem is to maximise

$$\frac{4}{9}\ln x_1 + \frac{2}{9}\ln x_2 + \frac{2}{9}\ln x_3 + \frac{1}{9}\ln x_4$$

subject to

$$
\begin{aligned}
4 &= \frac{16}{25}\left[\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right] \\
\Rightarrow 25 &= x_1 + x_2 + x_3 + x_4
\end{aligned}
$$

we use Lagrange multipliers.

$$\begin{aligned}
L(x_1, x_2, x_3, x_4) &= \frac{4}{9}\ln x_1 + \frac{2}{9}\ln x_2 + \frac{2}{9}\ln x_3 + \frac{1}{9}\ln x_4 - \lambda(x_1 + x_2 + x_3 + x_4 - 25) \\
\frac{\partial L}{\partial x_1} &= \frac{4}{9}\frac{1}{x_1} - \lambda \Rightarrow x_1 = \frac{4}{9\lambda} \\
\frac{\partial L}{\partial x_2} &= \frac{2}{9}\frac{1}{x_2} - \lambda \Rightarrow x_2 = \frac{2}{9\lambda} \\
\frac{\partial L}{\partial x_3} &= \frac{2}{9}\frac{1}{x_3} - \lambda \Rightarrow x_3 = \frac{2}{9\lambda} \\
\frac{\partial L}{\partial x_4} &= \frac{1}{9}\frac{1}{x_4} - \lambda \Rightarrow x_4 = \frac{1}{9\lambda}.
\end{aligned}$$

Since $25 = x_1 + x_2 + x_3 + x_4 = \frac{1}{\lambda}$, $\lambda = \frac{1}{25}$ and hence $x_1 = \frac{100}{9}$, $x_2 = \frac{50}{9} = x_3$ and $x_4 = \frac{25}{9}$. Now replicate!

In general, consider N-length words $\omega_1, \omega_2, \ldots, \omega_M$ ($M = 2^N$), measures $\mathbb{P}$ and $\mathbb{Q}$ with probabilities $p_1, p_2, \ldots, p_M$ and $q_1, q_2, \ldots, q_M$ respectively. x_i will represent $X(\omega_i)$. We then maximise

$$\sum_{i=1}^{M} p_i U(x_i)$$

subject to

$$\begin{aligned}
X_0 &= \frac{1}{(1+r)^N}\sum_{i=1}^{M} q_i x_i \\
&:= \sum_{i=1}^{M} p_i \zeta_i x_i
\end{aligned}$$

where

$$\zeta_i = \frac{1}{(1+r)^N}\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega_i).$$

Then

$$\begin{aligned}
L(x_i) &= \sum_{i=1}^{M} p_i U(x_i) - \lambda\left(\sum_{i=1}^{M} p_i \zeta_i x_i - X_0\right) \\
0 = \frac{\partial L}{\partial x_i} &= p_i U'(x_i) - \lambda p_i \zeta_i \\
0 &= U'(x_i) - \lambda \zeta_i.
\end{aligned}$$

This has a solution by the strict concavity of U!

3.4 Exercises

1. Shreve [2004] Exercise 3.4 p.84
2. (i) Verify that $U(x) = \frac{1}{p}x^p$, $0 < p < 1$ is a utility function.

   (ii) Repeat the utility algorithm of §3.3 with the ln-function replaced with this function.

Chapter 4

American Options

4.1 Introduction to American options

An American option is one that can be exercised at any time from the grant date (or rather, from the effective date) until the expiry date. An intermediate option is the Bermudan option, which allows for exercise at some specified set of nodes (in the discrete model) or some specified set of dates or date intervals (in the continuous model).

An American option has at least the worth of any Bermudan option, and this has at least the worth of the European option.

We will see that the discounted price of an American option is not a martingale (as it was in the European case) but is a supermartingale. Fortunately, this only occurs if the holder has failed to exercise optimally. While the option is worth more alive than dead, the discounted price process is a martingale.

As in the European case we find the cost of a hedge portfolio. The difficulty is that we need to prove that the hedge performs in the case of optimal behaviour by the holder of the option, but also in the case of suboptimal behaviour. In this case, we will show that the hedge portfolio is a super-hedge: it provides additional windfall income (exactly at those moments when the exercise behaviour is suboptimal). This suboptimal behaviour may either be failure to exercise optimally, or exercising too early.

4.2 Non-Path-Dependent American Derivatives

A non path-dependent American option is one whose payoff at time n with the stock price being S_n is $g(S_n)$ for some function g. Thus, we write $g(s)$, not $g(\omega_1\omega_2\ldots\omega_n)$.

We guess the following valuation algorithm (backwards recursion) for the value of the option:

$$V_N(s) = \max(g(s), 0)$$

$$V_n(s) = \max\left(g(s), \frac{1}{1+r}\left[q_H V_{n+1}(su) + q_T V_{n+1}(sd)\right]\right)$$

In the usual way we can calculate $V_0(S_0)$ and charge this for the option. We set up the hedge in the same way as before and delta hedge the option through its life. We need to show two things:

- that this hedge portfolio provides at least enough money to settle the option at any stage. In particular, if the holder exercises when the intrinsic value is lower than the continuation value, then the hedge generates an additional payoff for the writer.
- that any failure to exercise optimally (whatever that means) give positive income to the option writer.

We don't prove this theorem right now, because soon we will prove a more general result concerning path dependent options.

4.3 Stopping Times

The time at which an American option should be exercised is known as a stopping time. The stopping time will be set to ∞ if the option should be allowed to expire unexercised. Note that the stopping time must have a property which is similar to the property of a sequence of random variables being adapted.

Definition 4.3.1. A stopping time is a random variable τ that takes on the 'time values' $0,\ 1,\ \ldots,\ N$ or ∞, and satisfies the condition that if $\tau(\omega_1) = n$ and ω_1 and ω_2 are words that coincide on the first n letters, then $\tau(\omega_2) = n$.

Thus, stopping is based only on available information. If stopping occurs at time n, then this decision is based on the information available at time n, and not on the outcome of any subsequent events (thus, insider information on the stock price process is excluded).
We are aware that discounted American option prices are supermartingales. Now, we define a new process - the stopped discounted option price process.

Definition 4.3.2. Given a stochastic process Y and a stopping time τ, the stopped process corresponding to Y is the process which takes the value $Y_{n\wedge\tau}$ at time n.

The stopped discounted option price will be a martingale.

Example 4.3.3. Suppose $u = 2$, $d = 0.5$, $r = 0.25$, $N = 5$, $S = 100$. Suppose we have an American put option with strike 50.
Note that $q_H = q_T = 0.5$.
We value the option backwards through the tree as in Table 4.1.
Note that the discounted option price process is Markov, so we can represent it on the tree. However, the discounted stopped process is path dependent, so cannot be represented on a tree. For example, let Y denote the discounted option price process.
Suppose the path $HTTTH$ is observed. Then

- $Y_0 = 5.28$, $Y_1 = 1.28$, $Y_2 = 2.56$, $Y_3 = 5.12$, $Y_4 = 10.24$, $Y_5 = 0$.
- $Y_{0\wedge\tau} = 5.28$, $Y_{1\wedge\tau} = 1.28$, $Y_{2\wedge\tau} = 2.56$, $Y_{3\wedge\tau} = 5.12$, $Y_{4\wedge\tau} = 10.24$, $Y_{5\wedge\tau} = 10.24$.

Suppose the path $TTTHH$ is observed. Then

Spot	0	1	2	3	4	5
0	100	200	400	800	1600	3200
1		50	100	200	400	800
2			25	50	100	200
3				12.5	25	50
4					6.25	12.5
5						3.125
Option	0	1	2	3	4	5
0	5.28	1.6	0	0	0	0
1		11.6	4	0	0	0
2			25	10	0	0
3				37.5	25	0
4					43.75	37.5
5						46.875
Discounted option	0	1	2	3	4	5
0	5.28	1.28	0	0	0	0
1		9.28	2.56	0	0	0
2			16	5.12	0	0
3				19.2	10.24	0
4					17.92	12.288
5						15.36
Stop?	0	1	2	3	4	5
0	FALSE	FALSE	FALSE	FALSE	FALSE	FALSE
1		FALSE	FALSE	FALSE	FALSE	FALSE
2			TRUE	FALSE	FALSE	FALSE
3				TRUE	TRUE	FALSE
4					TRUE	TRUE
5						TRUE
Discounted expected hold value	0	1	2	3	4	5
0	5.28	1.28	0	0	0	0
1		9.28	2.56	0	0	0
2			12.16	5.12	0	0
3				14.08	6.144	0
4					13.824	12.288
5						15.36

Table 4.1: Pricing an American put option

- $Y_0 = 5.28$, $Y_1 = 9.28$, $Y_2 = 16$, $Y_3 = 19.2$, $Y_4 = 10.24$, $Y_5 = 0$.
- $Y_{0\wedge\tau} = 5.28$, $Y_{1\wedge\tau} = 9.28$, $Y_{2\wedge\tau} = 16$, $Y_{3\wedge\tau} = 16$, $Y_{4\wedge\tau} = 16$, $Y_{5\wedge\tau} = 16$.

The following theorem says that what we have observed holds generally:

Theorem 4.3.4 (Optional Sampling Theorem)**.** A supermartingale/martingale/submartingale stopped

at a stopping time is a supermartingale/martingale/submartingale.
The expected value of a stopped supermartingale/martingale/submartingale is greater than/equal to/less than the expected value of the process itself.

4.4 General American Derivatives

Definition 4.4.1. Suppose an American derivative has intrinsic value process G_n. This intrinsic value might be path dependent. The price process, W, is defined for $n = 0, 1, \ldots, N$ as

$$W_n(\omega_1 \ldots \omega_n) = \max_{\tau} \mathbb{E}_n^{\mathbb{Q}} \left[\mathbb{1}_{\{n \leq \tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - n}} \right] \tag{4.1}$$

The idea behind this definition is as follows: we have reached time n, the stopping times that might give us income henceforth are those in $\{n \leq \tau \leq N\}$. When the owner exercises according to a stopping time τ, the payoff will be at time τ and will be G_τ. The value thus should be the risk neutral discounted expectation of this payoff. The long party should choose to maximise over all stopping times.

Theorem 4.4.2. The price process satisfies:

(i) $W_n \geq \max\{G_n, 0\}$.

(ii) $\frac{1}{(1+r)^n} W_n$ a supermartingale.

(iii) W_n is the smallest process satisfying (i) and (ii).

Proof. (i) Let n be given. Suppose we take the stopping time τ which stops at n. Then the payoff is G_n. On the other hand, suppose we take the stopping time which stops at ∞. Then the payoff is 0. Thus the price is greater or equal to either of these.

(ii) Consider any stopping time τ. Clearly

$$W_n \geq \mathbb{E}_n^{\mathbb{Q}} \left[\mathbb{1}_{\{n+1 \leq \tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - n}} \right]$$

Now

$$\begin{aligned} \mathbb{E}_n^{\mathbb{Q}} \left[\mathbb{1}_{\{n+1 \leq \tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - n}} \right] &= \mathbb{E}_n^{\mathbb{Q}} \left[\frac{1}{1+r} \mathbb{1}_{\{n+1 \leq \tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - (n+1)}} \right] \\ &= \mathbb{E}_n^{\mathbb{Q}} \left[\frac{1}{1+r} \mathbb{E}_{n+1}^{\mathbb{Q}} \left[\mathbb{1}_{\{n+1 \leq \tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - (n+1)}} \right] \right] \end{aligned}$$

and so $W_n \geq \mathbb{E}_n^{\mathbb{Q}} \left[\frac{1}{1+r} W_{n+1} \right]$ as τ was arbitrary.

(iii) Let Y_n be such a process. Fix n and suppose τ does not stop before n. Then

$$\begin{aligned} \mathbb{1}_{\{\tau \leq N\}} G_\tau &\leq \mathbb{1}_{\{\tau \leq N\}} \max\{G_{\tau \wedge N}, 0\} + \mathbb{1}_{\{\tau = \infty\}} \max\{G_{\tau \wedge N}, 0\} \\ &= \max\{G_{\tau \wedge N}, 0\} \\ &\leq Y_{\tau \wedge N}. \end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}_n^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau\le N\}}\frac{G_\tau}{(1+r)^{\tau-n}}\right] &= (1+r)^n\mathbb{E}_n^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau\le N\}}\frac{G_\tau}{(1+r)^{\tau\wedge N}}\right]\\ &\le (1+r)^n\mathbb{E}_n^{\mathbb{Q}}\left[\frac{Y_{\tau\wedge N}}{(1+r)^{\tau\wedge N}}\right]\\ &\le (1+r)^n\frac{Y_{\tau\wedge n}}{(1+r)^{\tau\wedge n}}\\ &= Y_n\end{aligned}$$

because $\frac{Y_{\tau\wedge n}}{(1+r)^{\tau\wedge n}}$ is a supermartingale by the Optional Sampling Theorem. □

Theorem 4.4.3. Recursively define a process V_n

$$V_N(\omega_1\dots\omega_N) = \max\{G_N(\omega_1\dots\omega_N), 0\} \tag{4.2}$$

$$V_n(\omega_1\dots\omega_n) = \max\left\{G_n(\omega_1\dots\omega_n), \frac{1}{1+r}[q_HV_{n+1}(\omega_1\dots\omega_nH) + q_TV_{n+1}(\omega_1\dots\omega_nT)]\right\} \tag{4.3}$$

for $n < N$. Then

$$V_n(\omega_1\dots\omega_n) = W_n(\omega_1\dots\omega_n)$$

for all n and all part words $\omega_1\dots\omega_n$.

Proof. We show V_n satisfies properties (i) and (ii) in Theorem 4.4.2 and that V_n is the smallest process satisfying these properties.
First we prove the property (i) of Theorem 4.4.2. Clearly for $n = N$, from (4.2) property (i) holds. We proceed by induction backward in time. Suppose that for some n where $0 < n < N$ that

$$V_{n+1}(\omega_1\dots\omega_n\omega_{n+1}) \ge G_{n+1}(\omega_1\dots\omega_n\omega_{n+1}) \text{ and } V_{n+1}(\omega_1\dots\omega_n\omega_{n+1}) \ge 0$$

But, from (4.3) we have

$$V_n(\omega_1\dots\omega_n) \ge G_n(\omega_1\dots\omega_n)$$

and

$$V_n(\omega_1\dots\omega_n) \ge \frac{1}{1+r}[q_HV_{n+1}(\omega_1\dots\omega_nH) + q_TV_{n+1}(\omega_1\dots\omega_nT)]$$

Since $V_{n+1}(\omega_1\dots\omega_n\omega_{n+1}) \ge 0$, $q_H, q_T > 0$ and $\frac{1}{1+r} > 0$ it follows that

$$V_n(\omega_1\dots\omega_n) \ge 0$$

which completes the induction step.
From (4.3) it follows that

$$\begin{aligned}V_n(\omega_1\dots\omega_n) &\ge \frac{1}{1+r}[q_HV_{n+1}(\omega_1\dots\omega_n, H) + q_TV_{n+1}(\omega_1\dots\omega_nT)]\\ &= \mathbb{E}_n^{\mathbb{Q}}\left[\frac{1}{1+r}V_{n+1}\right](\omega_1\dots\omega_n)\end{aligned} \tag{4.4}$$

Multiplying both sides above with $\frac{1}{(1+r)^n}$ shows that $\frac{V_n}{(1+r)^n}$ a supermartingale.

Finally we need to show that $\frac{V_n}{(1+r)^n}$ is the smallest supermartingale satisfying properties (i) and (ii) in Theorem 4.4.2. Clearly, from (4.2) V_N is the smallest random variable such that $V_N \geq \max\{G_N, 0\}$. Again we proceed by backward induction in time. Suppose for some n with $0 < n < N$, that $\frac{V_{n+1}}{(1+r)^{n+1}}$ is the smallest supermartingale satisfying the properties (i) and (ii). The supermartingale property implies that V_n satisfies (4.4). Properties (i) and (ii) of Theorem 4.4.2 implies for $n = N-1, \dots, 0$

$$V_n(\omega_1 \dots \omega_n) \geq \max\left\{G_n(\omega_1 \dots \omega_n), \frac{1}{1+r}[q_H V_{n+1}(\omega_1 \dots \omega_n, H) + q_T V_{n+1}(\omega_1 \dots \omega_n T)]\right\}$$

But (4.3) shows that we have equality in the above so that $V_n(\omega_1 \dots \omega_n)$ is the smallest such process. Hence the induction step is complete. □

Theorem 4.4.4. Consider the N-period binomial model. For every n define the delta-hedging formula as

$$\Delta_n(\omega_1 \dots \omega_n) = \frac{V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)}{S_{n+1}(\omega_1 \dots \omega_n H) - S_{n+1}(\omega_1 \dots \omega_n T)}$$

and the consumption

$$C_n(\omega_1 \dots \omega_n) = V_n(\omega_1 \dots \omega_n) - \frac{1}{1+r}(q_H V_{n+1}(\omega_1 \dots \omega_n H) + q_T V_{n+1}(\omega_1 \dots \omega_n T))$$

Set $X_0 = V_0$ and define the replicating portfolio by forwards induction

$$\begin{aligned} X_{n+1}(\omega_1 \dots \omega_n \omega_{n+1}) &= \Delta_n(\omega_1 \dots \omega_n) S_{n+1}(\omega_1 \dots \omega_n \omega_{n+1}) \\ &\quad + (1+r)(X_n(\omega_1 \dots \omega_n) - C_n(\omega_1 \dots \omega_n) - \Delta_n(\omega_1 \dots \omega_n) S_n(\omega_1 \dots \omega_n)). \end{aligned}$$

Then

$$X_n(\omega_1 \dots \omega_n) = V_n(\omega_1 \dots \omega_n)$$

for all n and all part words $\omega_1 \dots \omega_n$. In particular, $X_n \geq G_n$ for all n.

Proof. From (4.3) it follows that

$$V_n(\omega_1 \dots \omega_n) \geq \frac{1}{1+r}[q_H V_{n+1}(\omega_1 \dots \omega_n H) + q_T V_{n+1}(\omega_1 \dots \omega_n T)]$$

and hence $C_n \geq 0$.
We proceed by forward induction on n.
The base case, $n = 0$, holds by assumption. Suppose the statement is true for n. First note that

$$V_n(\omega_1 \dots \omega_n) - C_n(\omega_1 \dots \omega_n) = \frac{1}{1+r}\left[q_H V_{n+1}(\omega_1 \dots \omega_n H) + q_T V_{n+1}(\omega_1 \dots \omega_n T)\right]$$

We now prove that $X_{n+1}(\omega_1 \dots \omega_n H) = V_{n+1}(\omega_1 \dots \omega_n H)$ holds:

$$\begin{aligned} &X_{n+1}(\omega_1 \dots \omega_n H) \\ &= \Delta_n(\omega_1 \dots \omega_n) u S(\omega_1 \dots \omega_n) + (1+r)\left[X_n(\omega_1 \dots \omega_n) - C_n(\omega_1 \dots \omega_n) - \Delta_n(\omega_1 \dots \omega_n) S(\omega_1 \dots \omega_n)\right] \\ &= (1+r) X_n(\omega_1 \dots \omega_n) + \Delta_n(\omega_1 \dots \omega_n) S(\omega_1 \dots \omega_n)\left[u - (1+r)\right] - (1+r) C_n(\omega_1 \dots \omega_n) \\ &= (1+r) V_n(\omega_1 \dots \omega_n) - (1+r) C_n(\omega_1 \dots \omega_n) + \left[V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)\right] q_T \\ &= q_H V_{n+1}(\omega_1 \dots \omega_n H) + q_T V_{n+1}(\omega_1 \dots \omega_n T) + \left[V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)\right] q_T \\ &= V_{n+1}(\omega_1 \dots \omega_n H). \end{aligned}$$

Similarly it can be shown that $X_{n+1}(\omega_1 \dots \omega_n T) = V_{n+1}(\omega_1 \dots \omega_n T)$.
Finally, since for all n $V_n \geq \max\{G_n, 0\}$ (property (i)) and $X_n = V_n$, it follows that $X_n \geq G_n$ for all n. □

Theorem 4.4.5 (Optimal Exercise)**.** The optimal strategy for the long party to an American option is to exercise the option the moment the intrinsic value equals the continuation value. That is, the stopping time τ maximising

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau \leq N\}} \frac{G_\tau}{(1+r)^\tau}\right]$$

is given by

$$\tau^* = \min\{n | V_n = G_n\}$$

Proof. First we show that

$$\frac{1}{(1+r)^{n\wedge\tau^*}} V_{n\wedge\tau^*}$$

is a martingale.
Suppose $\omega_1 \dots \omega_n$ is such that $\tau^* \geq n+1$. Then $V_n(\omega_1 \dots \omega_n) > G_n(\omega_1 \dots \omega_n)$ and from (4.3)

$$\begin{aligned} V_{n\wedge\tau^*}(\omega_1 \dots \omega_n) &= V_n(\omega_1 \dots \omega_n) \\ &= \frac{1}{1+r}[q_H V_{n+1}(\omega_1 \dots \omega_n H) + q_T V_{n+1}(\omega_1 \dots \omega_n T)] \\ &= \frac{1}{1+r}[q_H V_{(n+1)\wedge\tau^*}(\omega_1 \dots \omega_n H) + q_T V_{(n+1)\wedge\tau^*}(\omega_1 \dots \omega_n T)] \end{aligned}$$

which is the martingale property.
On the other hand, suppose that $\omega_1 \dots \omega_n$ is such that $\tau^* \leq n$, then

$$\begin{aligned} V_{n\wedge\tau^*}(\omega_1 \dots \omega_{\tau^*}) &= V_{\tau^*}(\omega_1 \dots \omega_{\tau^*}) \\ &= q_H V_{\tau^*}(\omega_1 \dots \omega_{\tau^*}) + q_T V_{\tau^*}(\omega_1 \dots \omega_{\tau^*}) \\ &= q_H V_{(n+1)\wedge\tau^*}(\omega_1 \dots \omega_n H) + q_T V_{\tau^*}(\omega_1 \dots \omega_n T) \end{aligned}$$

which again is the martingale property.
Since the stopped process $\frac{1}{(1+r)^{n\wedge\tau^*}} V_{n\wedge\tau^*}$ is a martingale

$$\begin{aligned} V_0 &= \mathbb{E}^{\mathbb{Q}}\left[\frac{V_{N\wedge\tau^*}}{(1+r)^{N\wedge\tau^*}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^* = \infty\}} \frac{V_N}{(1+r)^N}\right] \end{aligned} \tag{4.5}$$

For the paths that are such that $\tau^* = \infty$, $V_n > G_n$ for all n and in particular, $V_N > G_N$. But from (4.2) we have

$$V_N(\omega_1 \dots \omega_N) = \max\{G_N(\omega_1 \dots \omega_N), 0\}$$

which means that $G_N < 0$ and $V_N = 0$. Therefore, $\mathbb{1}_{\{\tau^* = \infty\}} V_N = 0$ and (4.5) becomes

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\tau^* \leq N} \frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right]$$

□

4.5 American Call Options

Theorem 4.5.1. Suppose a path-independent American option has a convex payoff function $g(s)$ satisfying $g(0) = 0$. Also suppose the stock pays no dividends. Then the option may as well be European.

Proof. Note that $g(\lambda s) \leq \lambda g(s)$ for $s \geq 0$, $\lambda \in [0, 1]$.
Then

$$\begin{aligned} g(S_n) &= g\left(\mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{1+r}\right]\right) \\ &\leq \mathbb{E}_n^{\mathbb{Q}}\left[g\left(\frac{S_{n+1}}{1+r}\right)\right] \\ &\leq \mathbb{E}_n^{\mathbb{Q}}\left[\frac{1}{1+r} g(S_{n+1})\right] \end{aligned}$$

so we have the submartingale property.
Applying the Optional Sampling theorem to $\frac{g(S_n)}{(1+r)^n}$ gives

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{g(S_{\tau\wedge N})}{(1+r)^{\tau\wedge N}}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{g(S_N)}{(1+r)^N}\right]$$

Thus for any stopping time τ,

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau\leq N\}}\frac{g(S_\tau)}{(1+r)^\tau}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{g(S_{N\wedge\tau})}{(1+r)^{N\wedge\tau}}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{g(S_N)}{(1+r)^N}\right]$$

which is the European value. As the stopping time was arbitrary, the American call is not greater in value than the European call. □

4.6 Exercises

1. Shreve [2004] Exercise 4.1 p.115
2. Shreve [2004] Exercise 4.2 p.115
3. Shreve [2004] Exercise 4.3 p.116
4. (Exam 2008) Consider a ten-period binomial model. Design a spreadsheet that enables one to price vanilla American calls or puts on a non-dividend paying stock. The initial stock price, the up and down factors, the per-period interest rate, the strike, and whether the option is a call or a put, are to be inputs.

   Show trees of the option value, the delta of the option at every node, and whether one should stop the option at any particular node.

5. (Exam 2008) Consider a ten period binomial model. Design a spreadsheet that enables one to price European derivatives whose payoff is $\max(S_{10} - S_8, 0)$.

   The initial stock price, the up and down factors, the per-period interest rate, and the strike, are to be inputs.

   Show trees of the option value and the delta of the option at every node.

HINT: using risk neutral valuation you can calculate the value of the option at time 8 for every node, because the only way it can end in the money from there is if the stock price increases at times 9 and 10. Thus from time 8 you can induct backwards in the tree just as normal, and find the deltas at times 7 back to 0 as normal.

The parts that remain is to work out the value at time 9, and the deltas at time 8 and 9.

You may wish to hand in your rough working in an exam book, but there is no requirement to do so.

6. (Exam 2010) Consider a ten-period binomial model. Design a spreadsheet that enables one to price the European derivative whose payoff is $S_{10} - S_9$. (It is a type of forward, there isn't an option to leave the derivative unexercised.)

   The initial stock price, the up and down factors, and the per-period interest rate are to be inputs.

   Show trees of the option value and the delta of the option at every node for $t = 0,\ 1,\ \ldots,\ 9$.

   What kind of hedge is in place here?

   Can you guess the value of the derivative at time 0, as a function of S and r?

7. (Exam 2008) Answer the questions on the theorem below.

   Theorem 4.6.1. Suppose a path-independent American option has a convex non-negative payoff function $g(s)$ satisfying $g(0) = 0$. Also suppose the stock pays no dividends. Then the option may as well be European.

   Proof. Note that $g(\lambda s) \leq \lambda g(s)$ for $s \geq 0$, $\lambda \in [0, 1]$.

   Then

$$g(S_n) = g\left(\mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{1+r}\right]\right) \tag{4.6}$$

$$\leq \mathbb{E}_n^{\mathbb{Q}}\left[g\left(\frac{S_{n+1}}{1+r}\right)\right] \tag{4.7}$$

$$\leq \mathbb{E}_n^{\mathbb{Q}}\left[\frac{1}{1+r}g(S_{n+1})\right] \tag{4.8}$$

   so we have the submartingale property.

   Thus for any stopping time τ,

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau \leq N\}}\frac{g(S_\tau)}{(1+r)^\tau}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{g(S_{N\wedge\tau})}{(1+r)^{N\wedge\tau}}\right] \tag{4.9}$$

$$\leq \mathbb{E}^{\mathbb{Q}}\left[\frac{g(S_N)}{(1+r)^N}\right] \tag{4.10}$$

   which is the European value. □

   (i) Why is $g(\lambda s) \leq \lambda g(s)$ for $s \geq 0$, $\lambda \in [0, 1]$?

   (ii) In equation (4.6), why do we have the equality $S_n = \mathbb{E}_n^{\mathbb{Q}}\left[\frac{S_{n+1}}{1+r}\right]$?

   (iii) Explain the inequality in (4.7).

(iv) Explain the inequality in (4.8).

(v) Explain the inequality in (4.10).

(vi) Why does (4.10) complete the proof?

(vii) Explain why the proof only applies to American calls and not to American puts.

Chapter 5

Random Walk

5.1 Introduction

Definition 5.1.1 (Random Walk)**.** Consider an infinite sequences of independent coin tosses of a fair coin. For such a sequence $\omega_1\omega_2\ldots$ and $j = 1,\ 2,\ \ldots$ the increments are defined as

$$X_j = \begin{cases} 1 & , \text{ if } \omega_j = H \\ -1 & , \text{ if } \omega_j = T \end{cases} \tag{5.1}$$

Let $M_0 = 0$ and define $M_n = \sum_{j=1}^{n} X_j = M_{n-1} + X_n$. Then M_n, $n = 0, 1, \ldots$ is called a random walk.

If $p_H = \frac{1}{2} = p_T$, then M_n is called a symmetric random walk.
For any fixed n, let $\#H$ be the number of heads and $\#T$ the number of tails by time n. Note that $\#H + \#T = n$, $\#H - \#T = M_n$. Let $N(n, m)$ be the number of paths that after n time steps are at level m. If such a path occurs, then $\#H + \#T = n$, $\#H - \#T = m$, thus $\#H = \frac{n+m}{2}$, $\#T = \frac{n-m}{2}$. Thus

$$N(n, m) = \binom{n}{\frac{n+m}{2}} = \frac{n!}{\frac{n+m}{2}!\,\frac{n-m}{2}!}$$

Each path has probability 2^{-n} of occurring.

5.2 First passage (hitting) times

Definition 5.2.1 (Hitting Time)**.** For this random walk, let τ_m be the first time that the path reaches the level m, that is,

$$\tau_m = \min\{n | M_n = m\}$$

This is a stopping time, called the first passage time.

Theorem 5.2.2. Let m be an integer. Then the symmetric random walk reaches m almost surely, that is,

$$\mathbb{P}(\tau_m < \infty) = 1$$

Proof. See the exercises. □

5.3 Reflection principle

What is the probability that $\tau_1 \leq 2j-1$? The set of paths that have reached level 1 by time $2j-1$ is equal to the (disjoint) union of

- the number of paths that are at 1 at time $2j-1$.
- the number of paths that are above 1 at time $2j-1$.
- the number of paths that are below 1 at time $2j-1$, but have hit 1 at some previous time.

By reflecting paths in the level 1, we see that the number of paths in the second and third item above are equal. Thus,

$$\begin{aligned}
\mathbb{P}\left[\tau_1 \leq 2j-1\right] &= \mathbb{P}\left[M_{2j-1} = 1\right] + 2\mathbb{P}\left[M_{2j-1} \geq 3\right] \\
&= \mathbb{P}\left[M_{2j-1} = 1\right] + \mathbb{P}\left[M_{2j-1} \geq 3\right] + \mathbb{P}\left[M_{2j-1} \leq -3\right] \\
&= 1 - \mathbb{P}\left[M_{2j-1} = -1\right]
\end{aligned}$$

The probability that $M_{2j-1} = -1$ is

$$\begin{aligned}
\tfrac{1}{2}^{2j-1} N(2j-1,-1) &= \tfrac{1}{2}^{2j-1} \binom{2j-1}{j} \\
&= \tfrac{1}{2}^{2j-1} \frac{(2j-1)!}{j!(j-1)!}
\end{aligned}$$

Finally, we can find the probability that $\tau_1 = 2j-1$.

$$\begin{aligned}
\mathbb{P}\left[\tau_1 = 2j-1\right] &= \mathbb{P}\left[\tau_1 \leq 2j-1\right] - \mathbb{P}\left[\tau_1 \leq 2j-3\right] \\
&= \mathbb{P}\left[M_{2j-3} = -1\right] - \mathbb{P}\left[M_{2j-1} = -1\right] \\
&= \tfrac{1}{2}^{2j-3} \frac{(2j-3)!}{(j-1)!(j-2)!} - \tfrac{1}{2}^{2j-1} \frac{(2j-1)!}{j!(j-1)!} \\
&= \tfrac{1}{2}^{2j-1} \frac{(2j-3)!}{j!(j-1)!} \left[4j(j-1) - (2j-1)(2j-2)\right] \\
&= \tfrac{1}{2}^{2j-1} \frac{(2j-3)!}{j!(j-1)!} (2j-2) \\
&= \tfrac{1}{2}^{2j-1} \frac{(2j-2)!}{j!(j-1)!}.
\end{aligned}$$

Finally,

$$
\begin{aligned}
\mathbb{E}[\tau_1] &= \sum_{j=1}^{\infty}(2j-1)\mathbb{P}[\tau_1 = 2j-1] \\
&= \sum_{j=1}^{\infty}(2j-1)\frac{1}{2}^{2j-1}\frac{(2j-2)!}{j!(j-1)!} \\
&= \sum_{j=1}^{\infty}\frac{1}{2}^{2j-1}\binom{2j-1}{j} \\
&> \sum_{j=1}^{3}\frac{1}{2}^{2j-1}\binom{2j-1}{j} + \sum_{j=4}^{\infty}\frac{1}{j} \\
&= \infty
\end{aligned}
$$

5.4 The perpetual American Put

Lemma 5.4.1. Fix m and consider the symmetric random walk. Then

$$
\mathbb{E}[e^{u\tau_m}] = \left(\frac{1-\sqrt{1-e^{2u}}}{e^u}\right)^{|m|}
$$

if $u \leq 0$. (That expectation is undefined if $u > 0$).

Proof. Assume $m > 0$ by symmetry. Since $\mathbb{P}[\tau_m < \infty] = 1$, for $\sigma > 0$ $\mathbb{E}[e^{\sigma m}\text{sech}^{\tau_m}(\sigma)] = 1$.
We now solve for $e^u = \alpha = \text{sech}(\sigma)$, obtaining $e^{-\sigma} = \frac{1\pm\sqrt{1-\alpha^2}}{\alpha}$. For $0 < \alpha < 1$, the positive root gives an invalid solution, so $e^{-\sigma} = \frac{1-\sqrt{1-\alpha^2}}{\alpha}$.
Now substitute. Note that $e^{\sigma m}$ is a constant, so it can be moved out of the expectation and transferred. □

A perpetual option is one that has no maturity date. In this section we study the American perpetual put.
Suppose we have our usual parameters, and suppose that the strike of the put is $K = S2^{-n}$.
It is clear that the value of this option is a function of spot price only, and not of time - the option has no theta. Furthermore, it is clear the stopping time will be the first time that the spot price hits some level. (Shreve [2004] spends a lot of time proving this point).
Let τ_m be the first time the stock price hits $S2^{-m}$. Let $V(\tau_m)$ be the value of the option if we decide to exercise at time τ_m - of course this only makes sense for $m > n$. What is the optimal value of m?

$$
\begin{aligned}
V(\tau_m) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{4}{5}^{\tau_m}(K - S2^{-m})\right] \\
&= (S2^{-n} - S2^{-m})\mathbb{E}^{\mathbb{Q}}\left[\frac{4}{5}^{\tau_m}\right] \\
&= (S2^{-n} - S2^{-m})\frac{1}{2}^{m} \\
&= S2^{-(m+n)} - S2^{-2m}
\end{aligned}
$$

using Lemma 5.4.1 with $\alpha = \frac{4}{5}$. This is maximised when $m = n + 1$. The value then is

$$\begin{aligned} V(\tau_{n+1}) &= S2^{-2(n+1)} \\ &= \frac{K^2}{4S} \end{aligned}$$

Thus the value of the put is

$$V = \begin{cases} K - S & \text{if } S \leq \frac{K}{2} \\ \frac{K^2}{4S} & \text{if } S \geq K \end{cases}$$

5.5 Exercises

1. Shreve [2004] Exercise 5.6 p.140
2. Shreve [2004] Exercise 5.7 p.140
3. Shreve [2004] Exercise 5.8 p.140
4. (Exam 2010) Answer the questions on the theorem below:

   Consider an infinite sequence of independent coin tosses of a fair coin. Let $M_0 = 0$ and define M_n to be the number of heads minus the number of tails by time n. (So M_n, $n = 0, 1, \ldots$ is a random walk.) Let $m \in \mathbb{N}$ be some level. Then the random walk M_n reaches m almost surely.

   Proof: Let $N(n, m)$ be the number of paths that after n time steps are at level m.

   Let τ_m be the first time that the path reaches the level m, that is,

   $$\tau_m = \min\{n | M_n = m\}$$

   Suppose for convenience that m is even, and let N be even. Then

   $$\begin{aligned}
   \mathbb{P}[M_n < m\, \forall n \leq N] &= \mathbb{P}[\tau_m > N] && (5.2)\\
   &= 1 - \mathbb{P}[\tau_m \leq N] && (5.3)\\
   &= 1 - \mathbb{P}[M_N = m] - 2\mathbb{P}[M_N \geq m + 2] && (5.4)\\
   &= 1 - \mathbb{P}[M_N = m] - \mathbb{P}[M_N \geq m + 2] - \mathbb{P}[M_N \leq -(m + 2)] && (5.5)\\
   &= \sum_{k=-\frac{m}{2}}^{\frac{m}{2}-1} \mathbb{P}[M_N = 2k] && (5.6)\\
   &= \left(\frac{1}{2}\right)^N \sum_{k=-\frac{m}{2}}^{\frac{m}{2}-1} N(N, 2k) && (5.7)\\
   &= \left(\frac{1}{2}\right)^N \sum_{k=-\frac{m}{2}}^{\frac{m}{2}-1} \binom{N}{\frac{N}{2} + k} && (5.8)\\
   &\leq \frac{mN}{2^N} && (5.9)\\
   &\to 0 \text{ as } N \to \infty && (5.10)
   \end{aligned}$$

(i) Explain (5.2) in words.

(ii) Carefully explain (5.4).

(iii) Carefully explain (5.5).

(iv) How does (5.6) follow?

(v) How does (5.7) follow?

(vi) How does (5.8) follow?

(vii) How many terms are in the sum? Hence explain (5.9).

(viii) How would you prove (5.10)?

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