Introduction to Continuous Time Stochastic Calculus

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These notes are a dense introduction to the subject of stochastic calculus in continuous time. The assumption is that many of the concepts have already been studied in discrete time, so having detailed motivation and many examples is not always entirely necessary.

For additional references one is referred to Etheridge [2002], Shreve [2004], Brzeźniak and Zastawniak [2005], Lamberton and Lapeyre [2008], Karatzas and Shreve [1988], Björk [1998].

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Chapter 1

Basic definitions

1.1 Stochastic Process

For fixed t, $X_t = X_t(\omega)$ is a random variable. For fixed ω, $X_t(\omega)$ is a sample path of $\{X_t\}_{t\geq 0}$.

1.2 What is equality of processes?

- Processes X, Y are indistinguishable if $\mathbb{P}\left(\{X_t = Y_t,\ \forall t \in [0,\infty)\}\right) = 1$.
- X, Y are modifications of each other if $\mathbb{P}(X_t = Y_t) = 1,\ \forall t \in [0,\infty)$.
- X, Y have the same finite dimensional distributions if for any $(t_1,\ \dots,\ t_n)$ with $t_i \in [0,\infty)$ the densities of the random vectors $(X_{t_1},\ \dots,\ X_{t_n})$, $(Y_{t_1},\ \dots,\ Y_{t_n})$ coincide.

1.3 Filtration

A filtration is a collection $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$.
$\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \geq 0)$ is the σ-algebra generated by all the $\mathcal{F}_t$.
A stochastic process is a family of random variables $\{X_t\}_{t\geq 0}$ defined on some $(\Omega, \mathcal{F}, \mathbb{P})$.
If X a stochastic process then $\mathcal{F}_t^X$ is defined to be the smallest σ-algebra for which all X_s, $s \leq t$ are measurable w.r.t. that algebra.
$\mathbb{F}^X = \{\mathcal{F}_t^X, t \geq 0\}$ is the corresponding filtration.
Given $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_{t^+} = \bigcap\limits_{h>0} \mathcal{F}_{t+h}$ and $\mathcal{F}_{t^-}$ is the σ-algebra generated by $\{\mathcal{F}_s, s < t\}$.
$\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous/left-continuous if $\mathcal{F} = \mathcal{F}_{t^+}/\mathcal{F} = \mathcal{F}_{t^-}$ for all $t \geq 0$.
$\{\mathcal{F}_t\}_{t\geq 0}$ is continuous if it is both left- and right-continuous.
A filtration satisfies the usual conditions if

- it is right-continuous and
- $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets with respect to $\mathbb{P}$.

1.4 Adapted process

A random variable $X_t(\omega)$ is adapted if it is $\mathcal{F}_t$-measurable.

1.5 Random Time

A random time is a random variable that is a map $\tau : \Omega \to [0, \infty]$.

1.6 Stopping Time

A stopping time (with respect to filtration $\{\mathcal{F}_t\}_{t\geq 0}$) is a random time such that for all $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.
τ is finite if $\tau < \infty$.
τ is bounded if there is a $K \in [0, \infty)$ such that $\tau(\omega) \leq K$.

1.7 Tower Property

Let X be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{H} \subseteq \mathcal{G}$ be any two sub-σ-algebras of $\mathcal{F}$, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}] \tag{1.1}$$

1.8 Jensen's Inequality

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and X a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[\phi(X)] < \infty$. Then, for any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$,

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}] \tag{1.2}$$

1.9 Martingale

$\{X_t\}_{t\geq 0}$ is a sub/super-martingale relative to $(\{\mathcal{F}_{t\geq 0}\}, \mathbb{P})$ if

- $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$.
- $\{X_t\}_{t\geq 0}$ is $\mathcal{F}_t$-adapted.
- $\mathbb{E}[X_t|\mathcal{F}_s] \gtreqqless X_s$ a.s. for all $0 \leq s \leq t$.

1.10 Local Martingale

An $\{\mathcal{F}_t\}$-adapted process $\{X_t\}_{0\leq t\leq T}$ is a local martingale with respect to $\{\mathcal{F}_t\}$ if there exists a sequence $\{\tau_n\}$ such that

- $$\tau_n \uparrow T \text{ a.s. as } n \to \infty.$$
- For each n, $X_{t\wedge\tau_n}$ is a martingale with respect to $\{\mathcal{F}_t\}_{0\leq t\leq T}$.

X is a continuous local martingale if X is uniformly integrable on $[0, \tau_n]$ for all n.

Chapter 2

Brownian Motion

2.1 Gaussian

A real-valued random variable X is Gaussian with mean μ and variance σ^2 if the density f_X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma}}, \; x \in \mathbb{R}.$$

The moment generating function of X is given by

$$\mathbb{E}\left[e^{tX}\right] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent, then

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

2.2 Brownian Motion

A stochastic process W on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is standard Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if

- $W_0 = 0$ a.s.
- $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $s \leq t$. We say that W is Markov, or has no memory.
- $W_t - W_s \sim \mathcal{N}(0, t-s)$. In particular, the variance of W_t is t.
- W is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and has continuous paths.

$\mathbb{C}\text{ov}[W_s, W_t] = s \wedge t$: suppose $s \leq t$ then

$$\begin{aligned}
\mathbb{C}\text{ov}[W_s, W_t] &= \mathbb{E}[(W_s - \mathbb{E}[W_s])(W_t - \mathbb{E}[W_t])] \\
&= \mathbb{E}[W_s B_t] \\
&= \mathbb{E}[W_s(W_t - W_s + W_s)] \\
&= \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2] \\
&= s^2 \text{ by the independence of Brownian increments.}
\end{aligned}$$

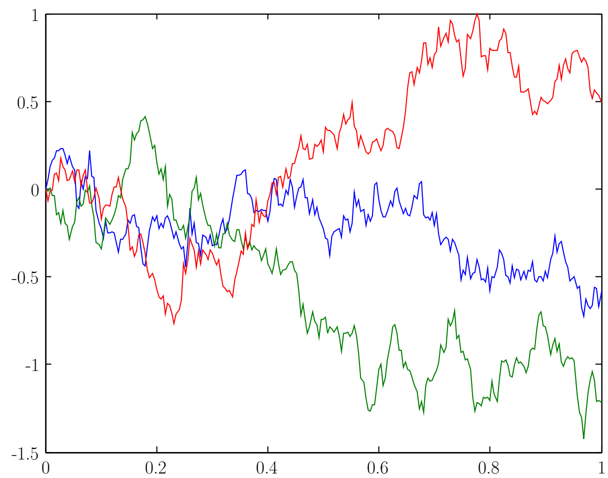

Figure 2.1: Three standard Brownian motion paths.

2.2.1 Brownian Motion is a Martingale

- For $t \geq 0$

$$\begin{aligned}\mathbb{E}[|W_t|] &= \int_{-\infty}^{\infty} \frac{|z|}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz \\ &= 2\int_{0}^{\infty} \frac{z}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz \\ &= -\sqrt{\frac{2t}{\pi}} \left[e^{-\frac{z^2}{2t}} \right]_0^{\infty} \\ &= \sqrt{\frac{2t}{\pi}} < \infty.\end{aligned}$$

- W is $\{\mathcal{F}_t\}_{t\geq 0}$ adapted by definition.
- For $0 \leq s \leq t$

$$\begin{aligned}\mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[(W_t - W_s) + W_s|\mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s] + W_s \\ &= W_s.\end{aligned}$$

2.3 $\{W_t^2 - t\}_{t\geq 0}$ is a Martingale

For $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}[W_t^2 - W_s^2|\mathcal{F}_s] &= \mathbb{E}[(W_t - W_s)^2 + 2W_s(W_t - W_s)|\mathcal{F}_s] \\
&= \mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + \mathbb{E}[2W_s(W_t - W_s)|\mathcal{F}_s] \\
&= \mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + 2W_s\mathbb{E}[(W_t - W_s)|\mathcal{F}_s] \\
&= \mathbb{E}[W_{t-s}^2|\mathcal{F}_s] \\
&= t - s.
\end{aligned}$$

2.4 Remark - Lévy's characterisation

If $X = \{X_t\}_{t\geq 0}$ is a continuous martingale and $X_t^2 - t$ is also a martingale, then X is a Brownian motion.

2.5 A Markov process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. An adapted process $\{X_t\}_{t\geq 0}$ is a Markov process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if for any bounded Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = g(X_s),\ 0 \leq s \leq t$$

for some Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$.

2.5.1 Brownian Motion is Markov

If $W = \{W_t\}_{t\geq 0}$ is standard Brownian motion, then W is Markov:
For $0 \leq s \leq t$

$$\mathbb{E}[f(W_t)|\mathcal{F}_s] = \mathbb{E}[f((W_t - W_s) + W_s)|\mathcal{F}_s].$$

Define

$$\begin{aligned}
g(x) &:= \mathbb{E}[f((W_t - W_s) + x)|\mathcal{F}_s] \\
&= \mathbb{E}[f((W_t - W_s) + x)] \text{ by independence.} \\
&= \int_{\mathbb{R}} f(y + x)\phi_{0,t-s}(y)dy
\end{aligned}$$

where $\phi_{0,t-s}$ is the density of $\mathcal{N}(0, t - s)$. Then

$$\mathbb{E}[f(W_t)|\mathcal{F}_s] = \mathbb{E}[f((W_t - W_s) + W_s)|\mathcal{F}_s] = g(W_s) \text{ a.s.}$$

2.6 Arithmetic Brownian motion

Let $\{W_t\}_{t\geq 0}$, $x_0 \in \mathbb{R}$, μ and σ are constants.

$$X_t = x_0 + \mu t + \sigma W_t,\ t \geq 0$$

is called Arithmetic Brownian motion.

2.7 Geometric Brownian motion

Let $\{W_t\}_{t\geq 0}$, $x_0 \in \mathbb{R}$, μ and σ are constants.

$$X_t = e^{x_0+\mu t+\sigma W_t},\ t \geq 0$$

is called geometric Brownian motion.

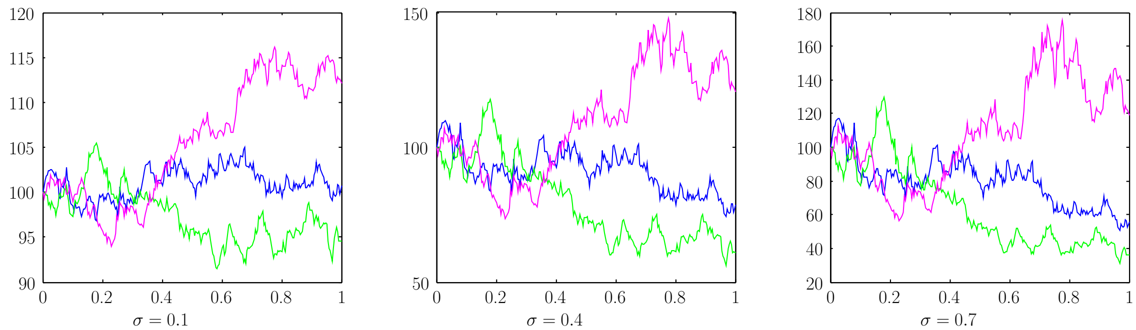

Figure 2.2: Three GBM paths for various values of σ.

2.8 Exercises

1. If X is normal, what is $\mathbb{E}[X]$, $\mathbb{E}\left[X^2\right]$, $\mathbb{E}\left[X^3\right]$, $\mathbb{E}\left[X^4\right]$?
2. Show that arithmetic Brownian motion is Markov.
3. Show that geometric Brownian motion is Markov.
4. Suppose W_t is a standard Brownian motion.
   (i) Find the distribution of $2W_2 + 3W_3 - 4W_4$.
   (ii) Calculate $\mathbb{E}[(W_4 - W_1)(W_3 - W_1)|W_1 = 2]$.
   (iii) Calculate $\mathbb{E}[(W_4 - W_2)(W_3 - W_2)|W_1 = 2]$.

Chapter 3

Total and Quadratic variation

3.1 Natural Filtration

For $t \geq 0$

$$\mathcal{F}_t^W = \sigma(W_u, 0 \leq u \leq t).$$

3.2 Transformations of Brownian Motion

Let $W = \{W_t, \mathcal{F}_t\}_{t\geq 0}$ be a standard Brownian motion, then the following processes are standard Brownian motion:

- Symmetry -

$$-W = \{-W_t, \mathcal{F}_t\}_{t\geq 0}.$$

- Scaling - for all $c > 0$,

$$X = \{X_t, \mathcal{F}_{ct}\}_{t\geq 0} \text{ defined by } X_t = \frac{1}{\sqrt{c}} W_{ct},\ t \geq 0.$$

- Time-homogeneity - for all $s > 0$, $Y = \{Y_t, \mathcal{F}_{t+s}\}_{t\geq 0}$ defined by $Y_t = W_{t+s} - W_t$, $t \geq 0$ independent of $\mathcal{F}_s$.

- Time-inversion - $Z = \left\{Z_t, \mathcal{F}_t^Z\right\}_{t\geq 0}$ defined by $Z_0 = 0$, $Z_t = tB_{1/t}$ for $t > 0$:

$$\begin{aligned} Z_t - Z_s &= tB_{1/t} - sB_{1/s} \\ &= (t-s)W_{1/t} - s(W_{1/s} - W_{1/t}) \\ &\sim \mathcal{N}\left(0, (t-s)^2 \frac{1}{t} + s^2 \left(\frac{1}{s} - \frac{1}{t}\right)\right) \\ &= \mathcal{N}(0, t-s) \end{aligned}$$

3.3 Brownian Motion is Nowhere Differentiable

$W = \{W_t, \mathcal{F}_t\}_{t\geq 0}$ is self-similar

$$W_{at} \stackrel{d}{=} \sqrt{a} W_t.$$

Self-similarity implies that the sample paths of Brownian motion are nowhere differentiable:

$$\frac{dB}{dt} = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{W_{t+\Delta t} - W_t}{\Delta t}.$$

Since $\Delta W = O\left(\sqrt{\Delta t}\right)$, $\frac{\Delta W}{\Delta t} = O\left(\frac{1}{\sqrt{\Delta t}}\right)$ so that $\frac{dB}{dt} \to \infty$ as $\Delta t \to 0$.

3.4 Wald's Martingale (an example of the Doléan's-Dade exponential)

Theorem 3.4.1. Let $\lambda \in \mathbb{R}$, then

$$M = \{M_t\}_{t\geq 0} = \left\{e^{\lambda W_t - \frac{1}{2}\lambda^2 t}\right\}_{t\geq 0}$$

is a martingale.

Proof.

$$\begin{aligned}
\mathbb{E}[|M_t|] &= \mathbb{E}[M_t] \\
&= \int_{-\infty}^{\infty} e^{\lambda u - \frac{1}{2}\lambda^2 t} \frac{e^{-\frac{u^2}{2t}}}{\sqrt{2\pi t}} du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u-\lambda t)^2}{2t}} \\
&= 1 < \infty.
\end{aligned}$$

$\{M_t\}_{t\geq 0}$ is clearly $\mathcal{F}_t$-adapted.

$$\begin{aligned}
\mathbb{E}\left[\frac{M_{t+s}}{M_t}\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[e^{\lambda(W_{t+s}-W_t) - \frac{1}{2}\lambda^2(t+s-t)}\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{\lambda W_s - \frac{1}{2}\lambda^2 s}\right] \\
&= \mathbb{E}[M_s] \\
&= 1.
\end{aligned}$$

□

3.5 First Hitting Time/First Passage Time

Let $a \in \mathbb{R}$ be given and fixed then

$$T_a = \inf\{s \geq 0 : W_s = a\}$$

T_a is a stopping time (technically this is quite hard to show).

Let $\{W_t\}_{t\geq 0}$ and T_a for constant $a \in \mathbb{R}$ be given. Then

$$\mathbb{E}\left[e^{-\lambda T_a}\right] = e^{-|a|\sqrt{2\lambda}}.$$

For $a \neq 0$, the density of T_a is given by

$$f_a(s) = \frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{\sqrt{2s}}},\ s > 0.$$

However, $\mathbb{E}[T_a] = \infty$.

Theorem 3.5.1 (Optional Sampling Theorem)**.** A martingale stopped at a stopping time is a martingale. A supermartingale (or submartingale) stopped at a stopping time is a supermartingale (or submartingale, respectively).

3.6 Bounded Variation

A function $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ if for some constant $M > 0$

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < M$$

for all partitions $a = x_0 < \ldots < x_n = b$ of $[a, b]$.

3.6.1 Total Bounded Variation

If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then the total variation of f on $[a, b]$ is

$$V_f(a, b) = \sup_{\Pi} \sum_{i=1}^{n(\pi)} |f(x_i) - f(x_{i-1})|$$

where Π is the class of all finite partitions $a = x_0 < \ldots < x_{n(\pi)} = b$ of $[a, b]$.
If $f : [a, b] \to \mathbb{R}$ is monotonically increasing then $V_f(a, b) = f(b) - f(a)$. If $f : [a, b] \to \mathbb{R}$ is continuous on $\mathbb{R}$ and differentiable on (a, b) with $\sup_{a<x<b} |f(x)| \leq M$ then by the mean-value theorem

$$\begin{aligned}
\sum_{i=1}^{n(\pi)} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^{n(\pi)} |f'(x_i)(x_i - x_{i-1})| \\
&\leq \sum_{i=1}^{n(\pi)} M(x_i - x_{i-1}) \\
&= M(b - a)
\end{aligned}$$

and so $V_f(a, b) \leq M(b - a)$.

3.7 Quadratic Variation

A function f has quadratic variation if over $[a, b]$ there exists an M such that

$$\sum_{i=1}^{n} [f(t_i) - f(t_{i-1})]^2 \leq M$$

for ALL partitions $a = x_0 < \ldots < x_n = b$ of $[a, b]$.

3.7.1 Total Quadratic Variation

The total quadratic variation of f on $[a, b]$ is $\lim_{\|\Pi\|\to 0} \sup_\Pi Q_\Pi$ where

$$Q_\Pi = \sum_{i=1}^{n(\pi)} [f(t_i) - f(t_{i-1})]^2$$

where Π is a finite partition $a = x_0 < \ldots < x_{n(\pi)} = b$ of $[a, b]$.
This is NOT the limit of a sequence of partitions, it is the limit of a net of partitions. Nets are a topological construct which have very nice properties very similar to those of sequences.
If $f \in \mathcal{C}^1[a, b]$ then its total quadratic variation is 0. So, quadratic variation is not a concept we see in ordinary calculus.

3.7.2 Quadratic Variation of a Continuous Stochastic Process

The quadratic variation of a stochastic process X on the interval $[0, t]$ is denoted $\langle X, X\rangle_t$.

Theorem 3.7.1. $\langle W, W\rangle_t = t$.

Proof. We need to show that $Q_\Pi \to t$ as $\|\Pi\| \to 0$. We shall show something slightly weaker: that $\mathbb{E}[Q_\Pi] = t$ and $\text{Var}[Q_\Pi] \to 0$. This actually proves that there is L^2-convergence of the relevant net of quadratic variations. It can be shown that this implies that there is a subnet which converges almost surely, and that allows us (after some more work) to make the desired conclusion.
Suppose $0 = t_0 < t_1 < \cdots < t_n = t$ is a partition of $[0, t]$. Let $W_i = W(t_i)$.

$$\begin{aligned}
\mathbb{E}[Q_\Pi] &= \mathbb{E}\left[\sum_{i=1}^{n} [W_i - W_{i-1}]^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[[W_i - W_{i-1}]^2\right] \\
&= \sum_{i=1}^{n} t_i - t_{i-1} \\
&= t
\end{aligned}$$

and

$$
\begin{aligned}
\operatorname{Var}[Q_\Pi] &= \mathbb{E}\left[Q_\Pi^2\right] - \mathbb{E}[Q_\Pi]^2 \\
&= \mathbb{E}\left[\sum_i \sum_j (W_i - W_{i-1})^2 (W_j - W_{j-1})^2\right] - t^2 \\
&= \sum_i \mathbb{E}\left[(W_i - W_{i-1})^4\right] + 2\sum_i \sum_{j>i} \mathbb{E}\left[(W_i - W_{i-1})^2\right] \mathbb{E}\left[(W_j - W_{j-1})^2\right] - t^2 \\
&= 3\sum_i (t_i - t_{i-1})^2 + 2\sum_i \sum_{j>i} (t_i - t_{i-1})(t_j - t_{j-1}) - t^2 \\
&= 3\sum_i (t_i - t_{i-1})^2 + t^2 - \sum_i (t_i - t_{i-1})^2 - t^2 \\
&= 2\sum_i (t_i - t_{i-1})^2 \\
&\le 2\|\Pi\| t \\
&\to 0 \text{ as } \|\Pi\| \to 0
\end{aligned}
$$

□

This result is informally denoted '$(dW)^2 = dt$'.

We now see that Lévy's characterisation of Brownian motion says that if $X = \{X_t\}_{t \ge 0}$ is a continuous martingale and $\langle X, X \rangle_t = t$ then X is a Brownian motion.

Proposition 3.7.2.

$$
\lim_{n\to\infty} \sum_{n=1}^{2^n} \left| W\left(\frac{k}{2^n}t\right) - W\left(\frac{k-1}{2^n}t\right) \right| = \infty \text{ a.s.}
$$

Proof.

$$
\sum_{n=1}^{2^n} \left| W\left(\frac{k}{2^n}t\right) - W\left(\frac{k-1}{2^n}t\right) \right| \ge \frac{\sum_{n=1}^{2^n} \left| W\left(\frac{k}{2^n}t\right) - W\left(\frac{k-1}{2^n}t\right) \right|^2}{\max_{j=1,\ 2,\ \dots,\ 2^n} \left| W\left(\frac{k}{2^n}t\right) - W\left(\frac{k-1}{2^n}t\right) \right|}
$$

The numerator on the right converges to t, while the denominator goes to 0 because Brownian paths are continuous and therefore uniformly continuous on bounded intervals. Hence the fraction on the right goes to infinity. □

3.8 Exercises

1. Let τ be the first time Brownian motion exits the open interval (a, b) where $a < 0 < b$.
   (i) Show that τ is a stopping time.
   (ii) What are the probabilities that exit occurs at a, and at b?
2. Draw an example, and then write down a formula, for a function that is continuous, but not of bounded variation.

3. Show that

$$\lim_{\|\Pi\|\to 0} \sum_i (W_i - W_{i-1})(t_i - t_{i-1}) = 0$$

$$\lim_{\|\Pi\|\to 0} \sum_i (t_i - t_{i-1})^2 = 0$$

Thus we now informally write $(dt)^2 = 0$, $dW\,dt = 0$. Thus write down a multiplication table:

$\times$	dW	dt
dW		
dt		

4. What is dW^3?

Chapter 4

The stochastic integral

4.1 Simple Predictable Process

A real-valued stochastic process $H = \{H_t\}_{t \geq 0}$ is called a simple predictable process if

$$H_t(\omega) = \phi_{-1} \mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} \phi_i(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

where $0 = t_0 < \ldots < t_n = T$ and ϕ_i is bounded $\mathcal{F}_{t_i}$-measurable random variable for $i = 0, \ldots, n-1$ and $\phi_{-1} \in \mathcal{F}_0$.
For fixed ω, H is a left-continuous step function of t.
If H is a simple predictable process then cH is a simple predictable process. If $H_1,\ H_2$ are simple predictable processes then $H_1 + H_2$ is a simple predictable process. $\mathcal{S}_T$ is the space for simple predictable processes.

4.2 $\{\mathcal{F}_t\}_{t \geq 0}$-Predictable

$\{X_t\}_{t \geq 0}$ is called $\{\mathcal{F}_t\}_{t \geq 0}$-predictable if X_t is $\mathcal{F}_{t^-}$-measurable for all $t \geq 0$.

4.3 Stochastic Integral of Simple Process

The stochastic integral of H is the continuous process $\{I(H)_t\}_{t \geq 0}$ for any $t \in (t_i, t_{i+1}]$

$$I(H)_t := \sum_{0 \leq i \leq k-1} \phi_i (W_{t_{i+1}} - W_{t_i}) + \phi_k (W_t - W_{t_k}) = \sum_{0 \leq i \leq k-1} \phi_i (W_{t_{i+1} \wedge t} - W_{t_i \wedge t})$$

which proves continuity of $t \mapsto I(H)_t$.
$W_0 = 0 \Rightarrow I(H)_0 = 0,\ \mathbb{P}$ a.s.
$I(\alpha H_1 + \beta H_2) = \alpha I(H_1) + \beta I(H_2)$ for any $\alpha,\ \beta \in \mathbb{R}$.

Theorem 4.3.1. $\left\{ \int_0^t H_s dB_s \right\}_{0 \leq t \leq T}$ is a continuous $\mathcal{F}_t$-martingale.

Proof. Show for any $t > s$ that

$$\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_u\right] = \int_0^s H_u dB_u.$$

Suppose $0 \leq t_j \leq s < t \leq t_{j+1} \leq T$ for some j. Then

$$\int_0^t H_u dB_u = \int_0^s H_u dB_u + \phi_j (W_t - W_s).$$

Since ϕ_j and $\int_0^s H_u dB_u$ are $\mathcal{F}_s$-measurable and $W_t - W_s$ is independent of $\mathcal{F}_s$ we have

$$\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_s\right] = \int_0^s H_u dB_u + \phi_j \mathbb{E}[W_t - W_s | \mathcal{F}_s] = \int_0^s H_u dB_u.$$

In particular if $0 \leq t_j \leq t \leq t_{j+1} \leq T$

$$\mathbb{E}\left[\int_0^{t_{j+1}} H_u dB_u | \mathcal{F}_t\right] = \int_0^t H_u dB_u \text{ and } \mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_{t_j}\right] = \int_0^{t_j} H_u dB_u.$$

Note that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_{t_{j-1}}\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_{t_j}\right] | \mathcal{F}_{t_{j-1}}\right] \\
&= \mathbb{E}\left[\int_0^{t_j} H_u dB_u | \mathcal{F}_{t_{j-1}}\right] \\
&= \int_0^{t_{j-1}} H_u dB_u.
\end{aligned}$$

By iteration for all $i \leq j$

$$\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_{t_i}\right] = \int_0^{t_i} H_u dB_u.$$

Finally, if $t_i \leq s < t_{i+1}$, $t_j \leq t < t_{j+1}$ then

$$\begin{aligned}
\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_0^t H_u dB_u | \mathcal{F}_{t_{i+1}}\right] | \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\int_0^{t_{j+1}} | \mathcal{F}_s\right] \\
&= \int_0^s H_u dB_s. \quad \square
\end{aligned}$$

Theorem 4.3.2. *Itô Isometry*

$$\mathbb{E}\left[\left(\int_0^t H_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 ds\right].$$

Proof.

$$H_t^2(\omega) = \phi_{-1}^2(\omega) \not\Vdash_{\{0\}}(t) + \sum_{i=0}^{n-1} \phi_i^2(\omega) \not\Vdash_{(t_i, t_{i+1}]}(t).$$

Therefore

$$\mathbb{E}\left[\int_0^t H_s^2 ds\right] = \sum_{i=0}^{n-1} \mathbb{E}[\phi_i^2](t_i - t_{i-1}).$$

Multiply terms in definition of $I(H)$ and observe that ϕ_i is independent of ΔW_i. $\square$

Corollary 4.3.3. For simple predictable processes H and K

$$\mathbb{E}\left[\left(\int_0^t H_s dB_s\right)\left(\int_0^t K_s dB_s\right)\right] = \mathbb{E}\left[\int_0^t H_s K_s ds\right]$$

and also

$$\mathbb{C}\text{ov}\left[\int_0^t H_u dB_u, \int_0^s K_u dB_u\right] = \mathbb{E}\left[\int_0^{t\wedge s} H_u K_u du\right].$$

4.4 Class $\mathcal{H} = \mathcal{H}[0,T]$

Extend Itô integral to class $\mathcal{H} = \mathcal{H}[0,T] = L^2_{ad}([0,T]\times\Omega)$ of processes $H : [0,T]\times\Omega \to \mathbb{R}$ to

- $(t,\omega) \mapsto H(t,\omega)$ is measurable with respect to $\mathcal{W}([0,T]) \otimes \mathcal{F}$.
- For each t, $\omega \mapsto H(t,\omega)$ is $\mathcal{F}_t$-measurable.
- $$\mathbb{E}\int_0^T H_t^2 dt = \int_0^T \mathbb{E}[H_t^2]dt < \infty.$$

4.5 Density Argument

Let $H \in \mathcal{H}[0,T]$. Then there exists a sequence of simple processes $H^{(n)}$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left(H_t - H_t^{(n)}\right)^2 dt\right] = 0.$$

Theorem 4.5.1. There exists a unique linear mapping J from $\mathcal{H}$ to the space of continuous $\mathcal{F}_t$-martingales defined on $[0,T]$ such that

- If $\{H_t\}_{0\le t\le T}$ is a simple predictable process

$\mathbb{P}$ a.s.

Let $H \in \widetilde{\mathcal{H}}[0,T]$, then

$$X = \left\{X_t := \int_0^t H_s dB_s\right\}_{0\le t\le T}$$

is a local martingale with respect to $\{\mathcal{F}_t\}_{0\le t\le T}$.

4.6 Exercises

1. Let $f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ -1 & \text{if } t > 1 \end{cases}$. Let $X_t = \int_0^t f(s)\, dW_s$.

   (i) Express X_t directly in terms of W_t.

   (ii) Verify that the process $X_t^2 - \int_0^t f^2(s)\, ds$ without invoking the Itô isometry i.e. explicitly prove the isometry in this special case.

2. Let W_t be standard Brownian motion and let $X_t = \int_0^t \text{sign}(W_s)\, dW_s$. Show that X_t is a Brownian motion.

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